The Strange Case of Dr. Admissibility and Mr. Derive

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Joint work with Noam Zeilberger and Robert Harper

Goal

A programming language that helps people:

- Define programming languages and logics
- Study their properties
- Use logics to reason about programs

Need

A programming language that makes it easy to:

- Represent syntax as data:
  - Programs of a programming language
  - Propositions of a logic
- Compute with these representations:
  - Write a compiler
  - Write a theorem prover

Derivability

In math

Polynomials over the reals:

$$f(x) = x^2 + 2x + 1$$

In math

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In math

Polynomials over the reals:

$$f(x) = x^2 + 2x + 1$$

Substitution: plug in for the variable

- $f(3) = 3^2 + 2*3 + 1$
- $f(y+5) = (y+5)^2 + 2(y+5) + 1$

In programming

Functional abstraction:

```
fact(x) = if (x = 0)
          then 1
          else  x * (fact (x-1))
```

In programming

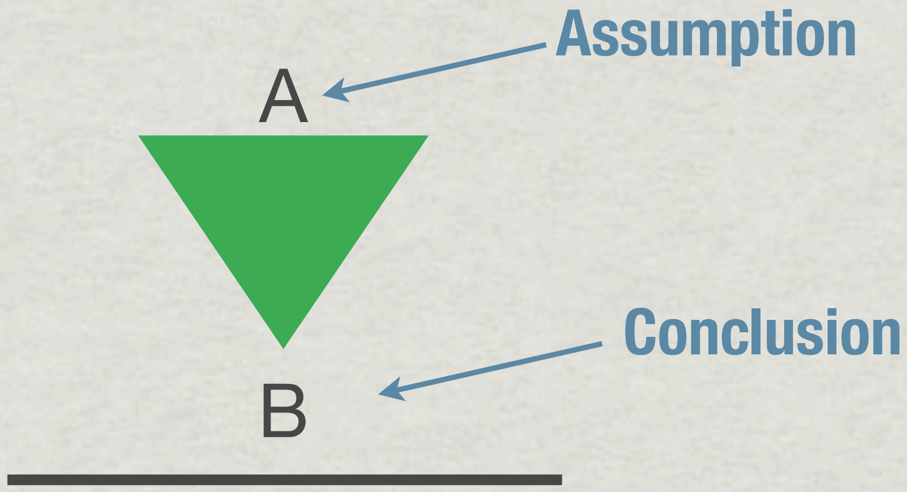
Functional abstraction:

```
fact(x) = if (x = 0)
          then 1
          else  x * (fact (x-1))
```

Compute by substitution:

```
fact(5) = if (5 = 0)
          then 1
          else  5 * (fact (5-1))
```

In logic



(A implies B)

In logic

If (A implies B) and A then B

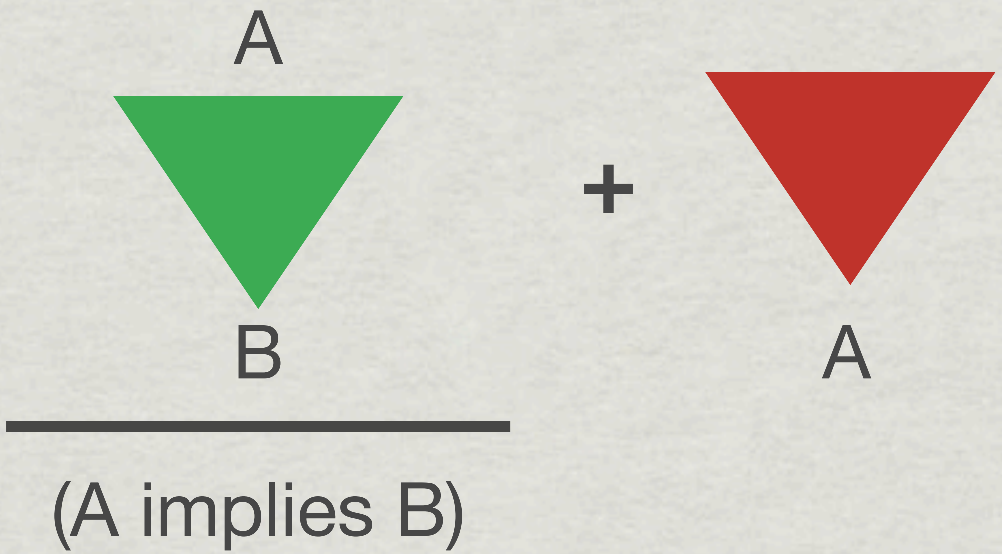
A

B

(A implies B)

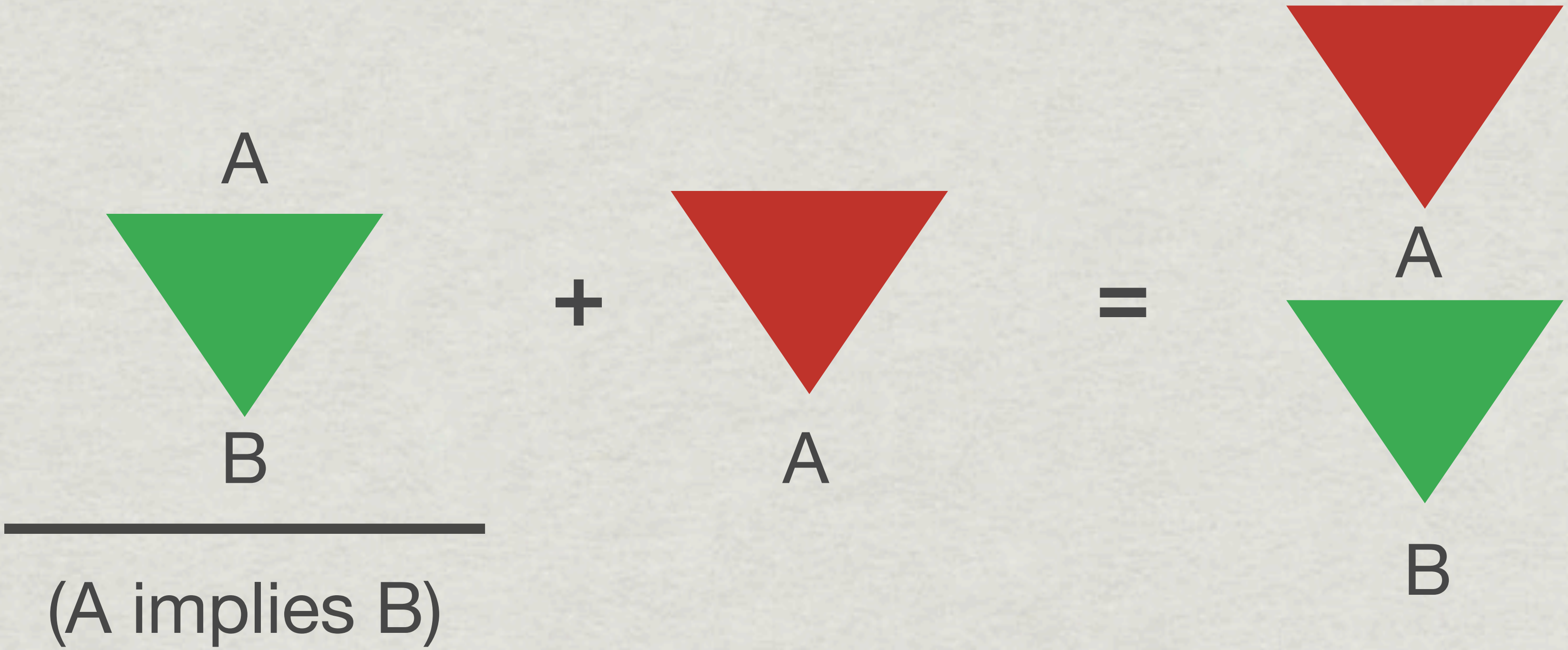
In logic

If (A implies B) and A then B



In logic

If (A implies B) and A then B



Derivability

Captures notion of ***terms-with-holes****,*
which can be filled by substitution

- Represent syntax with variable binding
- Represent hypothetical judgements (typing for a programming language)

Admissibility

In math

Function from reals to reals specified by:

- set of ordered pairs
- every number appears exactly once on the LHS

$$\{ (0, 1), (1, 4), (\sqrt{2}, 3 + 2\sqrt{2}), \ldots \}$$

In programming

Specify a function by its behavior (e.g., memo table):

Factorial = { (0 , 1),
(1 , 1),
(2 , 2),
(3 , 6),
(4 , 24),
…}

In logic

To prove $\forall x{:}nat.A(x)$,

prove $A(n)$ for each numeral n.

$A(0)$ and
$A(1)$ and
$A(2)$ and
$A(3)$ and
$A(4)$ ...

In logic

To prove $\forall x{:}nat.(even(x) \vee odd(x))$,

prove $(even(n) \vee odd(n))$ for each numeral n.

$even(0) \vee odd(0)$ and

$even(1) \vee odd(1)$ and

$even(2) \vee odd(2)$ and

$even(3) \vee odd(3)$ ...

Admissibility

*Captures notion of an **arbitrary transformation**:*
only i/o-behavior matters

- Represent proofs with infinitely many cases
- Compute with logical systems (compiler transformation, translation between logics)
- Prove theorems *about* logics

Derivability

$$f(x) = x^2 + 2x + 1$$

Admissibility

$$\{ (0, 1),\ (1, 4),\ (\sqrt{2}, 3 + 2\sqrt{2}),\ \ldots \}$$

Proof assistants

Coq

How well do existing proof assistants support derivability and admissibility?

Contributions

Definitions using

- Derivability: easy in Twelf, hard in Coq
- Admissibility: easy in Coq, hard in Twelf

Contributions

Definitions using

- Derivability: easy in Twelf, hard in Coq
- Admissibility: easy in Coq, hard in Twelf

This work:

Supports definitions using both derivability and admissibility

Outline

- Representations using derivability
- Representations using admissibility
- Mixed representations

Outline

* **Representations using derivability**
* Representations using admissibility
* Mixed representations

Derivability

How do we represent variable binding?

Derivability

How do we represent variable binding?

What is variable binding?

Syntax with binding

$f(x) = x^2 + 2x + 1$

$\forall x{:}nat.\,even(x) \lor odd(x)$

Structural properties

Properties of variables:

- Substitution
- Renaming
- Weakening

Structural properties

Properties of variables:

- Substitution (can plug in)
- Renaming
- Weakening

Structural properties

Properties of variables:

- Substitution (can plug in)
- **Renaming**
- Weakening

Dan's Homework

1. Factor the following: $f(x) = x^2 + 2x + 1$

Answer :

$$f(y) = (y + 1)^2$$

Dan's Homework

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$\cancel{f(y) = (y + 1)^2}$

The answer is
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Answer :

$\cancel{f(y) = (y + 1)^2}$

The answer is
$f(x) = (x + 1)^2$

(Dan finds a new math teacher...)

Renaming

Variables are **pronouns**:
pointer structure matters, names don't

$f(x) = x^2 + 2x + 1$ $\qquad\qquad$ $f(y) = y^2 + 2y + 1$

Renaming

Variables are **pronouns**:

pointer structure matters, names don't

$f(x,y) = x^2 + 2y + 1$ $\qquad$ $f(y,x) = y^2 + 2x + 1$

Renaming

Variables are **pronouns**:
pointer structure matters, names don't

$$f(x,y) = x^2 + 2y + 1 \qquad f(x,y) = y^2 + 2x + 1$$

Structural properties

Properties of variables:

- Substitution (can plug in)
- Renaming (names don't matter)
- **Weakening**

Weakening

"A polynomial in **one variable** is called a **univariate polynomial**, a polynomial in **more than one variable** is called a **multivariate polynomial** … when working with **univariate polynomials** one **does not exclude constant polynomials**...although strictly speaking **constant polynomials do not contain any variables** at all."

-- "Polynomial" in Wikipedia

Weakening

OK not to use a variable:

$$f(x) = 4$$

$$f(x,y) = x^2 + 2x + 1$$

Structural properties

Properties of variables:

- Substitution (can plug in)
- Renaming (names don't matter)
- Weakening (unused variables ok)

Representing derivability

Q: How to implement data with binding?

Answer 1: variables = strings

Polynomial: $f(x) = x^2 + 2 * x + 1$

Representation: ("x", "x"^{2} + 2 * "x" + 1)

name of bound variable **reference to binding**

Representing derivability

Q: How to implement data with binding?

Answer 1: variables = strings

$$(\text{“x”}, \text{“x”}^2 + 2 * \text{“x”} + 1)$$

Problems:

- Non-unique representations
- Do functions respect renaming?
- Have to implement substitution for each language

Representing derivability

Q: How to implement data with binding?

Other implement:

* de Bruijn indices/levels
* nominal logic [Pitts and Gabbay]
* cofinite quantification [Ayedemir et al.]

Representing derivability

Q: How to implement data with binding?

Other implement:

- de Bruijn indices/levels
- nominal logic [Pitts and Gabbay]
- cofinite quantification [Ayedemir et al.]

*These are **implementation**; we want the **interface**!*

Representing derivability

Key idea: represent binding with functions [HHP'93]

- Language provides a type of substitution functions with substitution, weakening, ...
- Implementation hidden from programmer

Representing derivability

f(x) = x + 3

represented by

λx.plus x 3

Substitution functions written λu.V, where
V is a **value** (e.g. **constructors** and variables)

Representing derivability

Constructors generate syntax:

- const : rational ⇒ exp

  Example: 3 represented by (const 3)

- plus : exp ⇒ exp ⇒ exp

  Example: 3 + 3 represented by

  plus (const 3) (const 3)

Representing derivability

Polynomials represented by substitution functions of type (exp ⇒ exp):

f(x) = x + 3

represented by

λx.plus x (const 3)

variables represented by variables!

Representing derivability

Polynomials represented by substitution functions of type (exp ⇒ exp):

substitution = application

f(x) = x + 3

represented by

λx.plus x (const 3)

f(5)

represented by

(λx.plus x (const 3)) applied to (const 5)

Representing derivability

Polynomials represented by substitution functions of type (exp ⇒ exp):

f(x) = x + 3

represented by

λx.plus x (const 3)

substitution = application

f(5)

represented by

(λx.plus x (const 3))
applied to (const 5)
=
plus (const 5) (const 3)

Adequacy

- Important that λu.V allows only substitution functions
- Cannot use admissibility to represent syntax: there are many more set-theoretic functions from expressions to expressions than polynomials

{(const n , 2 * 2 * 2 * … * 2) , …}

n times in total

Derivability

A theory of syntax with variable binding

1. Represent data with binding

    a) Define constructors (plus)

    b) Use derivability to represent binding ($\Rightarrow$)

2. Next: write recursive proofs/programs (admissibility)

Outline

- Representations using derivability
- **Representations using admissibility**
- Mixed representations

Admissibility

Admissibility (→) provided by pattern-matching functional programs

zero : nat
succ : nat ⇒ nat

double : nat → nat
double zero = zero
double (succ n) = succ (succ (double n))

Admissibility

Can pattern-match on substitution functions as well:

eval : (exp ⇒ exp) → rational → rational

eval (λx.V) r = evalexp ((λx.V) applied to (const r))

evalexp : exp → rational

evalexp (const n) = n

evalexp (plus n1 n2) =

add (evalexp n1) (evalexp n2)

apply a substitution function to perform substitution

Admissibility

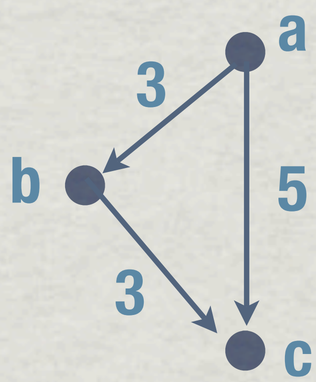
Twelf:

Can use admissibility to reason *about* logical systems, but not to *define* logical systems

Coq and our framework:

Can use admissibility in definitions

Admissibility



shortestPath(a_1,a_2,n) iff path(a_1,a_2,n) and
($\forall$m.path(a_1,a_2,m) $\rightarrow$ m $\geq$ n)

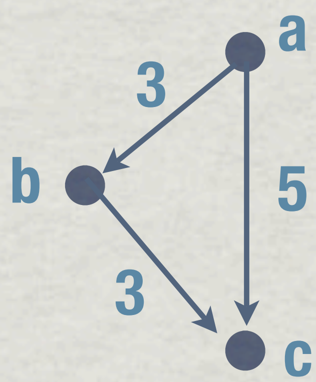
Admissibility

$\text{shortestPath}(a_1,a_2,n)$ iff $\text{path}(a_1,a_2,n)$ and
$(\forall m.\text{path}(a_1,a_2,m) \rightarrow m \geq n)$

can be written as

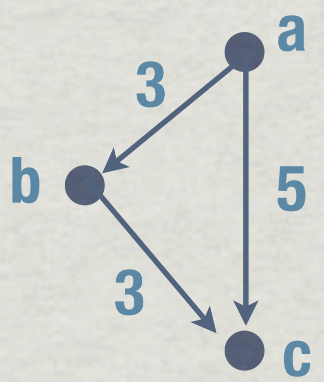
$$\frac{\text{path}(a_1,a_2,n) \qquad (\forall m.\text{path}(a_1,a_2,m) \rightarrow m \geq n)}{\text{shortestPath}(a_1,a_2,n)}$$

Admissibility



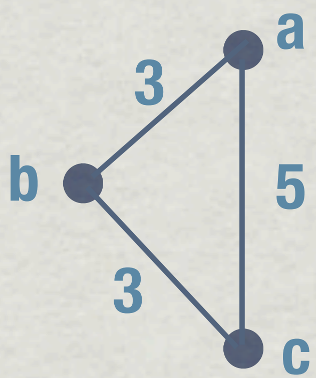
$$\frac{\text{path}(a,c,5) \qquad (\forall m.\text{path}(a,c,m) \rightarrow m \geq 5)}{\text{shortestPath}(a,c,5)}$$

Admissibility in rules



$$\frac{\text{path}(a,c,5) \qquad \begin{array}{l}\text{case [ac]: } 5 \geq 5 \\ \text{case [abc]: } 6 \geq 5\end{array}}{\text{shortestPath}(a,c,5)}$$

Admissibility in rules



(undirected, repeats ok)

case [ac]: $5 \geq 5$
case [abc]: $6 \geq 5$
case [acbc]: $11 \geq 5$

$$\frac{\text{path}(a,c,5) \qquad \ldots \text{ infinitely many more} \ldots}{\text{shortestPath}(a,c,5)}$$

Contrast with derivability

- Admissibility ($\forall$m.path(a,c,m) $\rightarrow$ m $\geq$ 5) allows proof by cases on the paths in this particular graph

- Derivability ($\forall$m.path(a,c,m) $\Rightarrow$ m $\geq$ 5) requires proof for a *new* path of *new* length m: **not true!**

Outline

- Representations using derivability
- Representations using admissibility
- **Mixed representations**

Arithmetic expressions

Arithmetic expressions with let-binding:

let x be (const 4) in (plus x x)

```
e ::= const n
    | let x be e1 in e2
    | plus e1 e2
    | times e1 e2
    | sub e1 e2
    | mod e1 e2
    | div e1 e2
```

Arithmetic expressions

Arithmetic expressions with let-binding:

let x be (const 4) in (plus x x)

e ::= const n
  | let x be e1 in e2
  | plus e1 e2
  | times e1 e2
  | sub e1 e2
  | mod e1 e2
  | div e1 e2

Suppose we want to treat binops uniformly…

Arithmetic expressions

Arithmetic expressions with let-binding

e ::= const n
    | let x be e1 in e2
    | binop e1 φ e2

where φ : (nat → nat → nat) is
the code for the binop.

Arithmetic expressions

const : nat ⇒ exp

let : exp ⇒ (exp ⇒ exp) ⇒ exp

binop : exp ⇒ (nat → nat → nat) ⇒ exp ⇒ exp

---

let x be (const 4) in (x + x)

represented by

let (const 4) (λx.binop x add x)

where add:(nat → nat → nat) is the code for addition

Arithmetic expressions

const : nat ⇒ exp

let : exp ⇒ (exp ⇒ exp) ⇒ exp

binop : exp ⇒ (nat → nat → nat) ⇒ exp ⇒ exp

---

let x be (const 4) in (x + x)

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Arithmetic expressions

const : nat ⇒ exp

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Arithmetic expressions

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Structural properties

Structural properties

Weakening for mixed representations:

Structural properties

Weakening for mixed representations:

1. Go from (nat $\Rightarrow$ nat) to (nat $\Rightarrow$ nat $\Rightarrow$ nat)

Structural properties

Weakening for mixed representations:

1. Go from (nat ⇒ nat) to (nat ⇒ nat ⇒ nat)

   λx.(λy.V)   [x doesn't occur in V]

Structural properties

Weakening for mixed representations:

1. Go from (nat ⇒ nat) to (nat ⇒ nat ⇒ nat)

   λx.(λy.V)   [x doesn't occur in V]

2. Go from (nat → nat) to (nat → nat → nat)

Structural properties

Weakening for mixed representations:

1. Go from (nat $\Rightarrow$ nat) to (nat $\Rightarrow$ nat $\Rightarrow$ nat)

   $\lambda x.(\lambda y.V)$ [x doesn't occur in V]

2. Go from (nat $\rightarrow$ nat) to (nat $\rightarrow$ nat $\rightarrow$ nat)

   drop x y = f y [x doesn't occur in f]

Structural properties

Structural properties

Weakening for mixed representations:

Structural properties

Weakening for mixed representations:

3. Go from (nat $\Rightarrow$ nat) to (nat $\rightarrow$ nat $\Rightarrow$ nat)

Structural properties

Weakening for mixed representations:

3. Go from (nat $\Rightarrow$ nat) to (nat $\rightarrow$ nat $\Rightarrow$ nat)

   h x = λy.V   [x doesn't occur in V]

Structural properties

Weakening for mixed representations:

3. Go from (nat $\Rightarrow$ nat) to (nat $\rightarrow$ nat $\Rightarrow$ nat)

   h x = λy.V   [x doesn't occur in V]

4. Go from (nat $\rightarrow$ nat) to (nat $\Rightarrow$ (nat $\rightarrow$ nat))

Structural properties

A function of type (nat → nat):

double zero = zero
double (succ n) = succ (succ (double n))

Structural properties

A (nat ⇒ (nat → nat)):

λx.double zero = zero
double (succ n) = succ (succ (double n))
double x = ???

Problem: double doesn't have a case for x,
so mixed representations not necessarily structural!

Structural properties

Our solution:

- λx.V used by pattern-matching; application (=substitution) is not built-in
- Nothing forces ⇒ to be structural
- Weakening/substitution implemented generically for a wide class of rule systems, using some technical conditions

Structural properties

const : nat $\Rightarrow$ exp

let : exp $\Rightarrow$ (exp $\Rightarrow$ exp) $\Rightarrow$ exp

binop : exp $\Rightarrow$ (nat $\rightarrow$ nat $\rightarrow$ nat) $\Rightarrow$ exp $\Rightarrow$ exp

- Can't weaken exp with nat:
  could need new case for $\rightarrow$ in a binop
- Can weaken exp with exp:
  doesn't appear to left of $\rightarrow$

Conclusion

Summary

- Derivability (variable binding) provided by substitution functions ($\Rightarrow$)
- Admissibility (arbitrary reasoning) provided by pattern-matching functional programs ($\rightarrow$)
- Our framework makes it easy to use both

Future work

- This talk: syntax of logical systems
  Need **dependent types** to represent judgements
- More examples mixing $\Rightarrow$/$\rightarrow$
- Thesis work: a language in which you can define logics and use them to reason about programs

Strange Case of

Dr. Admissibility

and

Mr. Derive