A 2-categorical framework for substructural and modal logics

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Linear Logic

$$\frac{\$1 \vdash \text{pepsi} \qquad \$1 \vdash \text{candy}}{\$1 \vdash \text{pepsi} \land \text{candy}}$$

Linear Logic

$$\cancel{\dfrac{\$1 \vdash \text{pepsi} \qquad \$1 \vdash \text{candy}}{\$1 \vdash \text{pepsi} \wedge \text{candy}}}$$

Linear Logic

$$\cancel{\dfrac{\$1 \vdash \text{pepsi} \qquad \$1 \vdash \text{candy}}{\$1 \vdash \text{pepsi} \wedge \text{candy}}}$$

$$\dfrac{\$1 \vdash \text{pepsi} \qquad \$1 \vdash \text{candy}}{\$1, \$1 \vdash \text{pepsi} \otimes \text{candy}}$$

Structural Rules

Weakening
$$\frac{\Gamma \vdash B}{\Gamma, x{:}A \vdash B}$$

Exchange
$$\frac{\Gamma, y{:}B, x{:}A \vdash B}{\Gamma, x{:}A, y{:}B \vdash C}$$

Contraction
$$\frac{\Gamma, x{:}A, y{:}A \vdash B}{\Gamma, x{:}A \vdash B}$$

A Pattern

- Operation on contexts Γ, with explicit or admissible structural properties
- Type constructor that “internalizes” the context operation, inherits the structural properties

Linear Logic

$$\frac{\Gamma \vdash A \qquad \Delta \vdash B}{\Gamma, \Delta \vdash A \otimes B} \qquad\qquad \frac{\Gamma, A, B \vdash C}{\Gamma, A \otimes B \vdash C}$$

Structural Properties

$$\frac{\Gamma \vdash A \qquad \Delta \vdash B}{\Gamma, \Delta \vdash A \otimes B} \qquad \frac{\Gamma, A, B \vdash C}{\Gamma, A \otimes B \vdash C}$$

$$\frac{\Gamma_0 \equiv \Gamma, \Delta \qquad \Gamma \vdash A \qquad \Delta \vdash B}{\Gamma_0 \vdash A \otimes B} \qquad \frac{\Gamma, A, B, \Delta \vdash C}{\Gamma, A \otimes B, \Delta \vdash C}$$

Structural Properties

$$\frac{\Gamma \vdash A \qquad \Delta \vdash B}{\Gamma, \Delta \vdash A \otimes B} \qquad\qquad \frac{\Gamma, A, B \vdash C}{\Gamma, A \otimes B \vdash C}$$

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lists up to permutation:
exchange, not weakening and contraction

Type inherits properties

$$\dfrac{\dfrac{A,B \equiv B,A \qquad B \vdash B \quad A \vdash A}{A,B \vdash B \otimes A}}{A \otimes B \vdash B \otimes A}$$

Bunched Implication

$A \wedge B$: A and B hold for the same resource

$A * B$: A and B hold for separate parts of the resource

[O'Hearn&Pym]

Bunched Implication

$A \wedge B$: A and B hold for the same resource

$A * B$: A and B hold for separate parts of the resource

$$A \wedge (B * C) \ncong (A \wedge B) * (A \wedge C)$$

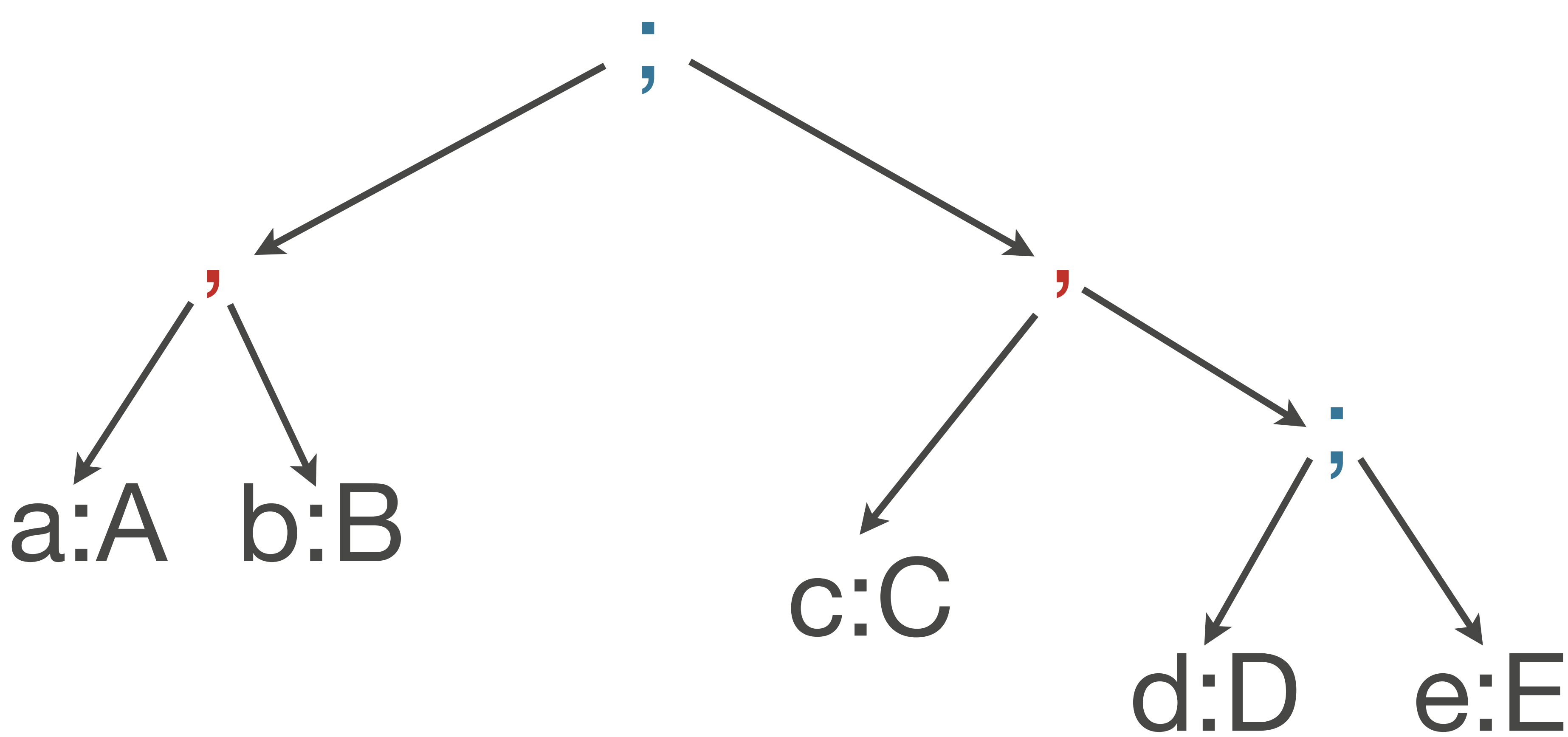
[O'Hearn&Pym]

Bunched Implication

$$(a{:}A,b{:}B);(c{:}C,(d{:}D;e{:}E)) \vdash F$$

Bunched Implication

(a:A,b:B);(c:C,(d:D;e:E)) ⊢ F



Bunched Implication

$$\frac{\Gamma_0 \equiv \Gamma;\Delta \quad \Gamma \vdash A \quad \Delta \vdash B}{\Gamma_0 \vdash A \wedge B} \qquad \frac{\Gamma[A;B] \vdash C}{\Gamma[A \wedge B] \vdash C}$$

$$\frac{\Gamma_0 \equiv \Gamma,\Delta \quad \Gamma \vdash A \quad \Delta \vdash B}{\Gamma_0 \vdash A * B} \qquad \frac{\Gamma[A,B] \vdash C}{\Gamma[A * B] \vdash C}$$

Bunched Implication

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, and ; are commutative monoids

Bunched Implication

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, and ; are commutative monoids

; has weakening and contraction

, doesn't

Linear Logic Exponentials

$$\frac{\Gamma\,;\ \cdot\ \vdash A}{\Gamma\,;\ \cdot\ \vdash\ !A}$$

$$\frac{\Gamma, A\,;\ \Delta \vdash C}{\Gamma;\ \Delta, !A \vdash C}$$

$$\frac{\Gamma, A;\ \Delta, A \vdash C}{\Gamma, A;\ \Delta \vdash C}$$

Linear Logic Exponentials

$$\frac{\Gamma \,;\, \cdot \vdash A}{\Gamma \,;\, \cdot \vdash !A}$$

$$\frac{\Gamma, A \,;\, \Delta \vdash C}{\Gamma ;\, \Delta, !A \vdash C}$$

exchange

$$\frac{\Gamma, A ;\, \Delta, A \vdash C}{\Gamma, A ;\, \Delta \vdash C}$$

Linear Logic Exponentials

$$\frac{\Gamma \,;\, \cdot \vdash A}{\Gamma \,;\, \cdot \vdash\, !A}$$

$$\frac{\Gamma, A \,;\, \Delta \vdash C}{\Gamma ;\, \Delta, !A \vdash C}$$

plus weakening, contraction

exchange

$$\frac{\Gamma, A;\, \Delta, A \vdash C}{\Gamma, A;\, \Delta \vdash C}$$

Pfenning-Davies S4

$$\frac{\Gamma \; ; \; \cdot \; \vdash A}{\Gamma \; ; \Delta \vdash \Box A}$$

$$\frac{\Gamma, A \; ; \; \Delta, \Box A \vdash C}{\Gamma ; \; \Delta, \Box A \; \vdash \; C}$$

$$\frac{\Gamma, A ; \; \Delta, A \; \vdash \; C}{\Gamma, A ; \; \Delta \; \vdash \; C}$$

Pfenning-Davies S4

$$\frac{\Gamma \,;\, \cdot \vdash A}{\Gamma \,;\, \Delta \vdash \Box A} \qquad \frac{\Gamma, A \,;\, \Delta, \Box A \vdash C}{\Gamma ; \Delta, \Box A \vdash C}$$

$$\frac{\Gamma, A; \Delta, A \vdash C}{\Gamma, A; \Delta \vdash C}$$

both cartesian

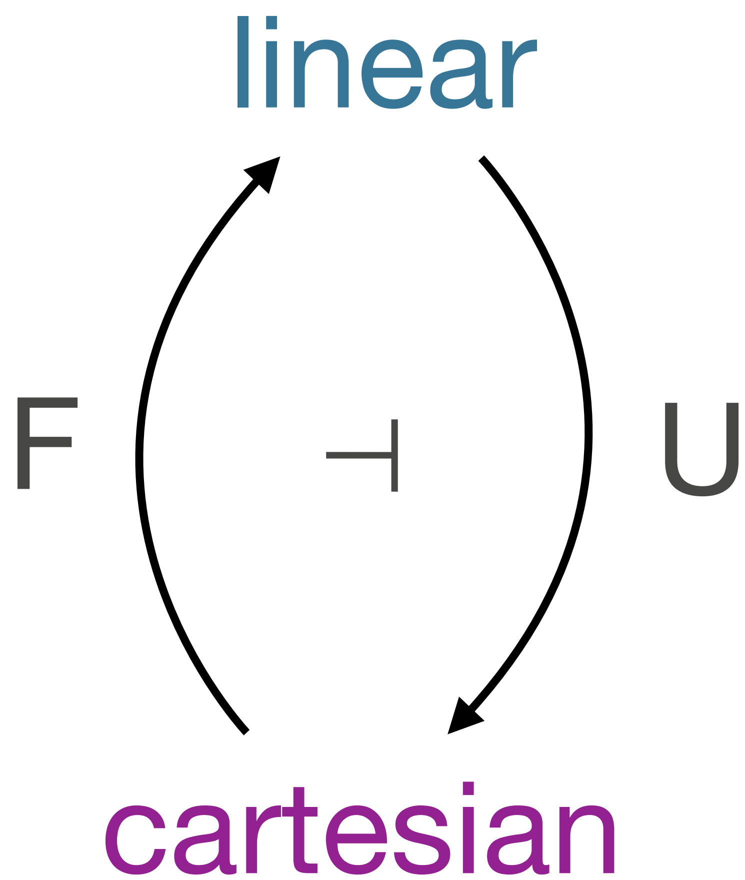
Adjoint logic

$$\begin{tikzcd} \text{linear} \arrow[d, bend left=60, "U"] \arrow[d, phantom, "\dashv"] \\ \text{cartesian} \arrow[u, bend left=60, "F"] \end{tikzcd}$$

[Benton&Wadler]

Adjoint logic

$$A ::= F\,C \mid A \otimes B \mid \ldots$$
$$C ::= U\,A \mid C \times D \mid \ldots$$



[Benton&Wadler]

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$$\frac{\Gamma \vdash_c C}{\Gamma\ ;\ \cdot \vdash_\ell F\ C}$$

$$\frac{\Gamma, C\ ;\ \Delta \vdash_\ell B}{\Gamma; \Delta, F\ C \vdash_\ell B}$$

[Benton&Wadler]

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$$\frac{\Gamma \vdash_c C}{\Gamma ; \cdot \vdash_\ell F\ C} \qquad \frac{\Gamma ; \cdot \vdash_\ell A}{\Gamma \vdash_c U\ A}$$

$$\frac{\Gamma, C ; \Delta \vdash_\ell B}{\Gamma ; \Delta, F\ C \vdash_\ell B} \qquad \frac{\Gamma ; \Delta, A \vdash_\ell B}{\Gamma, U\ A ; \Delta \vdash_\ell B}$$

[Benton&Wadler]

Adjoint logic

$$A ::= F\,C \mid A \otimes B \mid \ldots$$
$$C ::= U\,A \mid C \times D \mid \ldots$$

$$\begin{tikzcd}
\text{linear} \arrow[d, bend left, "U"] \\
\text{cartesian} \arrow[u, bend left, "F"] \arrow[u, phantom, "\dashv"]
\end{tikzcd}$$

$$!A := F U\,A$$

$$\frac{\Gamma \vdash_c C}{\Gamma\,;\,\cdot \vdash_\ell F\,C} \qquad \frac{\Gamma; \cdot \vdash_\ell A}{\Gamma \vdash_c U\,A}$$

$$\frac{\Gamma, C\,;\,\Delta \vdash_\ell B}{\Gamma; \Delta, F\,C \vdash_\ell B} \qquad \frac{\Gamma;\Delta, A \vdash_\ell B}{\Gamma, U\,A;\Delta \vdash_\ell B}$$

[Benton&Wadler]

Adjoint logic

$$\begin{tikzcd} \text{truth} \arrow[d, bend left=60, "U"] \arrow[d, phantom, "\dashv"] \\ \text{validity} \arrow[u, bend left=60, "F"] \end{tikzcd}$$

$$\Box A := F\,U\,A$$

$$\frac{\Gamma \vdash_{v} C}{\Gamma ; \Delta \vdash_{t} F\,C} \qquad \frac{\Gamma ; \cdot \vdash_{t} A}{\Gamma \vdash_{v} U\,A}$$

$$\frac{\Gamma, C ; \Delta \vdash_{t} B}{\Gamma ; \Delta, F\,C \vdash_{t} B} \qquad \frac{\Gamma ; \Delta, A \vdash_{t} B}{\Gamma, U\,A ; \Delta \vdash_{t} B}$$

[Reed]

Structural Properties

$$\cdot\ ; \mathrm{F}\ \mathrm{C} \otimes \mathrm{F}\ \mathrm{D} \vdash_{\ell} \mathrm{F}\ (\mathrm{C} \times \mathrm{D})$$

Structural Properties

$$\frac{\cdot\,;\ F\ C,\ F\ D \vdash_{\ell} F\ (C \times D)}{\cdot\ ;\ F\ C \otimes F\ D \vdash_{\ell} F\ (C \times D)}$$

Structural Properties

$$\dfrac{\dfrac{C \,;\, F\,D \vdash_{\ell} F\,(C \times D)}{\cdot \,;\, F\,C, F\,D \vdash_{\ell} F\,(C \times D)}}{\cdot \,;\, F\,C \otimes F\,D \vdash_{\ell} F\,(C \times D)}$$

Structural Properties

$$\dfrac{\dfrac{\dfrac{C, D \,;\, \cdot \vdash_\ell F\,(C \times D)}{C \,;\, F\,D \vdash_\ell F\,(C \times D)}}{\cdot \,;\, F\,C, F\,D \vdash_\ell F\,(C \times D)}}{\cdot \,;\, F\,C \otimes F\,D \vdash_\ell F\,(C \times D)}$$

Structural Properties

$$
\dfrac{\dfrac{\dfrac{\dfrac{C, D \vdash_{c} C \times D}{C, D \,;\, \cdot \vdash_{\ell} F\,(C \times D)}}{C \,;\, F\,D \vdash_{\ell} F\,(C \times D)}}{\cdot \,;\, F\,C,\, F\,D \vdash_{\ell} F\,(C \times D)}}{\cdot \,;\, F\,C \otimes F\,D \vdash_{\ell} F\,(C \times D)}
$$

Structural Properties

$$\dfrac{\dfrac{\dfrac{\dfrac{C, D \vdash_{c} C \times D}{C, D \,;\, \cdot \vdash_{\ell} F\,(C \times D)}}{C \,;\, F\,D \vdash_{\ell} F\,(C \times D)}}{\cdot \,;\, F\,C, F\,D \vdash_{\ell} F\,(C \times D)}}{\cdot \,;\, F\,C \otimes F\,D \vdash_{\ell} F\,(C \times D)}$$

(but not all left adjoints are monoidal functors)

Cohesive HoTT

$$\begin{tikzcd}
\text{Spaces} \arrow[d, bend right=70, "\int"'] \arrow[d, "\Gamma"] \\
\text{Sets} \arrow[u, bend left=35, "\dashv\ \Delta"] \arrow[u, bend right=45, "\dashv\ \nabla"']
\end{tikzcd}$$

$$\int \dashv \Delta \dashv \Gamma \dashv \nabla$$

A Pattern

$$\otimes \quad \mathbf{!} \quad \wedge_* \quad \square \quad \mathbf{F}$$

- Operation on contexts Γ, with structural properties
- Type constructor that “internalizes” the context operation, inherits the structural properties

Today

Develop a logic in which
$\otimes$ ***!*** $\wedge_*$ $\Box$ ***F are all instances of one connective***

Today

Develop a logic in which

$\otimes$ ***!*** $\wedge_*$ $\Box$ ***F are all instances of one connective***

$\multimap$ $\rightarrow$ $-_*$ ***U are all instances of another***

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Why?

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Develop a logic in which

$\otimes$! $\wedge_*$ $\Box$ F are all instances of one connective

$\multimap$ $\rightarrow$ $-_*$ U are all instances of another

Why?

- Pattern to abstraction

Today

Develop a logic in which

$\otimes$! $\wedge_*$ $\Box$ F are all instances of one connective

$\multimap$ $\rightarrow$ $-_*$ U are all instances of another

Why?

- Pattern to abstraction
- Tool for studying new situations

Today

Develop a logic in which
$\otimes$! $\wedge_{*}$ $\square$ F are all instances of one connective
$\multimap$ $\rightarrow$ -$*$ U are all instances of another

Why?

- Pattern to abstraction
- Tool for studying new situations

$$
\begin{tikzcd}[column sep=large, row sep=huge]
\text{Spaces} \arrow[d, bend right=80, "\int"'] \arrow[d, "\Gamma" description] \\
\text{Sets} \arrow[u, bend left=40, "\Delta"] \arrow[u, bend right=80, "\nabla"']
\end{tikzcd}
$$

$$\int \dashv \Delta \dashv \Gamma \dashv \nabla$$

Judgements

Sequent	$\Gamma\ [\alpha] \vdash A$
Context description	$\psi \vdash \alpha : p$

Context Descriptions

Modes	$p, q, \ldots$
Context Descriptions	$x_1{:}p_1, \ldots, x_n{:}p_n \vdash \alpha : q$
Structural Properties	$\alpha \Rightarrow \beta$

Context Descriptions

Modes	$p, q, \ldots$
Context Descriptions	$x_1{:}p_1, \ldots, x_n{:}p_n \vdash \alpha : q$
Structural Properties	$\alpha \Rightarrow \beta$

- Types p,q are “modes”/kinds of types/contexts

Context Descriptions

Modes	p,q,…
Context Descriptions	$x_1{:}p_1, \ldots, x_n{:}p_n \vdash \alpha : q$
Structural Properties	$\alpha \Rightarrow \beta$

- Types p,q are “modes”/kinds of types/contexts
- Terms α are descriptions of the context

Context Descriptions

Modes	p,q,…
Context Descriptions	$x_1{:}p_1, \ldots, x_n{:}p_n \vdash \alpha : q$
Structural Properties	$\alpha \Rightarrow \beta$

- Types p,q are “modes”/kinds of types/contexts
- Terms α are descriptions of the context
- “Reductions” $\alpha \Rightarrow \beta$ are structural properties

Context Descriptions

cartesian

Modes	p,q,…
Context Descriptions	$x_1{:}p_1, \ldots, x_n{:}p_n \vdash \alpha : q$
Structural Properties	$\alpha \Rightarrow \beta$

- Types p,q are "modes"/kinds of types/contexts
- Terms α are descriptions of the context
- "Reductions" $\alpha \Rightarrow \beta$ are structural properties

Examples

Linear $x{:}I,\ y{:}I \vdash x \otimes y : I$

$a{:}A, b{:}B, c{:}C, d{:}D \vdash \ldots$ $(a \otimes b) \otimes (c \otimes d)$

Examples

BI

$$x{:}b,\ y{:}b \vdash x * y : b$$

$$x{:}b,\ y{:}b \vdash x \wedge y : b$$

$(a{:}A, b{:}B)\ ;\ (c{:}C, (d{:}D ; e{:}E)) \vdash \ldots$ $(a * b) \wedge (c * (d \wedge e))$

Examples

Adjoint

$$\begin{tikzcd} \color{teal}{\text{linear}} \arrow[d, bend left=60, "U"] \arrow[d, phantom, "\dashv"] \\ \color{purple}{\text{cartesian}} \arrow[u, bend left=60, "F"] \end{tikzcd}$$

$$x{:}l,\ y{:}l \vdash x \otimes y : l$$

$$x{:}c,\ y{:}c \vdash x \times y : c$$

$$y{:}c \vdash f(y) : l$$

$$x{:}A, y{:}B \ ;\ z{:}C, w{:}D \vdash \ldots$$

$$f(x \times y) \otimes (z \otimes w)$$

Examples

Linear logic $x{:}I,\ y{:}I \vdash x \otimes y : I$ $\quad \cdot \vdash 1 : I$

Structural Rules $x \otimes (y \otimes z) \cong (x \otimes y) \otimes z$ $\quad x \otimes 1 \cong x$

$x \otimes y \cong y \otimes x$ $\quad 1 \otimes x \cong x$

Examples

Cartesian logic $x{:}I,\ y{:}I \vdash x \times y : I$ $\quad \cdot \vdash 1 : I$

Structural Rules $x \times (y \times z) \cong (x \times y) \times z$ $\quad x \times 1 \cong x$

$x \times y \cong y \times x$ $\quad 1 \times x \cong x$

Examples

Cartesian logic $x{:}I,\ y{:}I \vdash x \times y : I$ $\quad \cdot \vdash 1 : I$

Structural Rules $x \times (y \times z) \cong (x \times y) \times z$ $\quad x \times 1 \cong x$

$x \times y \cong y \times x$ $\quad 1 \times x \cong x$

$x \Rightarrow 1$ **weakening**

Examples

Cartesian logic $x{:}I,\ y{:}I \vdash x \times y : I$ $\qquad \cdot \vdash 1 : I$

Structural Rules $x \times (y \times z) \cong (x \times y) \times z$ $\qquad x \times 1 \cong x$

$x \times y \cong y \times x$ $\qquad 1 \times x \cong x$

$x \Rightarrow 1$ **weakening**

$y \Rightarrow y \times y$ **contraction**

Judgements

$A \ \mathsf{type}_p$

Judgements

$A ::= F\,C \mid A \otimes B \mid \ldots$

$C ::= U\,A \mid C \times D \mid \ldots$

$$A \ \mathrm{type}_p$$

Judgements

$A \ \mathsf{type}_p$

Judgements

$$A\ \mathrm{type}_p$$

$$x_1{:}A_1 \ldots x_n{:}A_n\ [\alpha] \vdash A$$

Judgements

$$A \ \mathrm{type}_{p}$$

$$x_1{:}A_1 \ \ldots \ x_n{:}A_n \ [\alpha] \vdash A$$

$$x_1{:}p_1 \ \ldots \ x_n{:}p_n \vdash \alpha : p$$

Judgements

$$A \ \mathrm{type}_p$$

$$x_1{:}A_1 \ \ldots \ x_n{:}A_n \ [\alpha] \vdash A$$

$$x_1{:}p_1 \ \ldots \ x_n{:}p_n \vdash \alpha : p$$

Hypothesis

$$\frac{\beta \Rightarrow x \qquad x : P \in \Gamma}{\Gamma\,[\beta] \vdash P}$$

Hypothesis

up to whatever structural properties you've asserted,
what you need to use is x

$$\frac{\beta \Rightarrow x \qquad x : P \in \Gamma}{\Gamma\,[\beta] \vdash P}$$

Structural Rules (Admiss)

$$\frac{\Gamma\ [\alpha] \vdash A \qquad \Gamma, x{:}A\ [\beta] \vdash B}{\Gamma\ [\ \beta[\alpha/x]\ ] \vdash B}
\qquad
\frac{\begin{array}{l}\Gamma\ [\alpha] \vdash A \\ \beta \Rightarrow \alpha\end{array}}{\Gamma\ [\beta] \vdash A}$$

Structural Rules (Admiss)

$$\frac{\Gamma\ [\alpha] \vdash A \qquad \Gamma, x{:}A\ [\beta] \vdash B}{\Gamma\ [\ \beta[\alpha/x]\ ] \vdash B} \qquad \frac{\begin{array}{c}\Gamma\ [\alpha] \vdash A \\ \beta \Rightarrow \alpha\end{array}}{\Gamma\ [\beta] \vdash A}$$

$$\frac{\Gamma\ [\ \alpha\ ] \vdash B}{\Gamma, x{:}A\ [\ \alpha\ ] \vdash B} \qquad \frac{\Gamma, x{:}A, y{:}A\ [\alpha] \vdash B}{\Gamma, x{:}A\ [\alpha[y/x]] \vdash B}$$

$$\frac{\Gamma, y{:}B, x{:}A\ [\alpha] \vdash B}{\Gamma, x{:}A, y{:}B\ [\alpha] \vdash C}$$

F (functor) types

internalize context descriptions as types

$$\frac{\psi \vdash \alpha : p \qquad \Delta\ \mathrm{ctx}_\psi}{F_\alpha\ \Delta\ \mathrm{type}_p}$$

e.g. $A \otimes B := F_{(x \otimes y)}\ (x{:}A,\ y{:}B)$

F Left

$$\frac{\Gamma, A, B, \Delta \vdash C}{\Gamma, A \otimes B, \Delta \vdash C}$$

$$\frac{\Gamma, A \,;\, \Delta \vdash C}{\Gamma;\, \Delta, !A \vdash C}$$

$$\frac{\Gamma[A;B] \vdash C}{\Gamma[A \wedge B] \vdash C}$$

$$\frac{\Gamma[A,B] \vdash C}{\Gamma[A * B] \vdash C}$$

F Left

$$\frac{\Gamma, \Delta, \Gamma' \; [\, \beta[\alpha/x] \,] \vdash B}{\Gamma,\ x{:}F_{\alpha}(\Delta),\ \Gamma' \; [\beta] \vdash B}$$

F Left

remember where in the tree Δ variables occur

$$\frac{\Gamma,\Delta,\Gamma' \; [\; \beta[\alpha/x] \;] \vdash B}{\Gamma,\; x{:}F_\alpha(\Delta),\; \Gamma' \; [\beta] \vdash B}$$

F Right

$$\frac{\Gamma_0 \equiv \Gamma, \Delta \quad \Gamma \vdash A \quad \Delta \vdash B}{\Gamma_0 \vdash A \otimes B}$$

$$\frac{\Gamma_0 \equiv (\Gamma; \cdot) \quad \Gamma ; \cdot \vdash A}{\Gamma_0 \vdash !A}$$

F Right

$$\frac{\beta \Rightarrow \alpha[\gamma] \qquad \Gamma\,[\gamma] \vdash \Delta}{\Gamma\,[\beta] \vdash F_\alpha\, \Delta}$$

Exchange

$$x : F_{(y \otimes z)}(y{:}A, z{:}B)\ [x] \vdash F_{(z' \otimes y')}(y'{:}A, z'{:}B)$$

Exchange

$$\frac{y{:}A, z{:}B\ [y \otimes z] \vdash F_{(z' \otimes y')}(y'{:}A, z'{:}B)}{x : F_{(y \otimes z)}(y{:}A, z{:}B)\ [x] \vdash F_{(z' \otimes y')}(y'{:}A, z'{:}B)}$$

Exchange

$$\frac{\overline{\quad y{:}A, z{:}B\ [y \otimes z] \vdash F_{(z' \otimes y')}(y'{:}A, z'{:}B)\quad}}{x : F_{(y \otimes z)}(y{:}A, z{:}B)\ [x] \vdash F_{(z' \otimes y')}(y'{:}A, z'{:}B)}$$

Exchange

$$y \otimes z \Rightarrow (z' \otimes y')[?]$$

$$\dfrac{\overline{\; y{:}A, z{:}B\ [y \otimes z] \vdash F_{(z' \otimes y')}(y'{:}A, z'{:}B) \;}}{x : F_{(y \otimes z)}(y{:}A, z{:}B)\ [x] \vdash F_{(z' \otimes y')}(y'{:}A, z'{:}B)}$$

Exchange

$$\dfrac{\dfrac{}{y{:}A,z{:}B\ [y\otimes z] \vdash F_{(z'\otimes y')}(y'{:}A,z'{:}B)}}{x : F_{(y\otimes z)}(y{:}A,z{:}B)\ [x] \vdash F_{(z'\otimes y')}(y'{:}A,z'{:}B)}$$

Exchange

$$y \otimes z \Rightarrow (z' \otimes y')[z/z', y/y']$$

$$\dfrac{\dfrac{}{y{:}A, z{:}B\ [y \otimes z] \vdash F_{(z' \otimes y')}(y'{:}A, z'{:}B)}}{x : F_{(y \otimes z)}(y{:}A, z{:}B)\ [x] \vdash F_{(z' \otimes y')}(y'{:}A, z'{:}B)}$$

Exchange

$$\frac{\dfrac{}{y{:}A, z{:}B\ [y \otimes z] \vdash F_{(z' \otimes y')}(y'{:}A, z'{:}B)}}{x : F_{(y \otimes z)}(y{:}A, z{:}B)\ [x] \vdash F_{(z' \otimes y')}(y'{:}A, z'{:}B)}$$

Exchange

$$y \otimes z \Rightarrow z \otimes y$$

$$\frac{\overline{y{:}A, z{:}B\ [y \otimes z] \vdash F_{(z' \otimes y')}(y'{:}A, z'{:}B)}}{x : F_{(y \otimes z)}(y{:}A, z{:}B)\ [x] \vdash F_{(z' \otimes y')}(y'{:}A, z'{:}B)}$$

Exchange

$$\frac{\begin{array}{c} y \otimes z \Rightarrow z \otimes y \\ y{:}A[y] \vdash A \end{array}}{\dfrac{y{:}A, z{:}B\ [y \otimes z] \vdash F_{(z' \otimes y')}(y'{:}A, z'{:}B)}{x : F_{(y \otimes z)}(y{:}A, z{:}B)\ [x] \vdash F_{(z' \otimes y')}(y'{:}A, z'{:}B)}}$$

Exchange

$$y \otimes z \Rightarrow z \otimes y$$

$$y{:}A[y] \vdash A$$

$$z{:}B[z] \vdash B$$

$$\frac{\dfrac{y \otimes z \Rightarrow z \otimes y \qquad y{:}A[y] \vdash A \qquad z{:}B[z] \vdash B}{y{:}A, z{:}B\ [y \otimes z] \vdash F_{(z' \otimes y')}(y'{:}A, z'{:}B)}}{x : F_{(y \otimes z)}(y{:}A, z{:}B)\ [x] \vdash F_{(z' \otimes y')}(y'{:}A, z'{:}B)}$$

Structural Properties

$$\begin{array}{c} A,B \vdash_{c} A \times B \\ \hline A,B \,;\, \cdot \vdash_{\ell} F\,(A \times B) \\ \hline A \,;\, F\,B \vdash_{\ell} F\,(A \times B) \\ \hline \cdot\,;\, F\,A,\, F\,B \vdash_{\ell} F\,(A \times B) \\ \hline \cdot \,;\, F\,A \otimes F\,B \vdash_{\ell} F\,(A \times B) \end{array}$$

(but not all left adjoints are monoidal functors)

$$x{:}\ F_f\, A \otimes F_f\, B\ [x] \vdash F_f\, (A \times B)$$

$$A \otimes B := F_{y \otimes z}(y{:}A, z{:}B) \qquad A \times B := F_{y \times z}(y{:}A, z{:}B)$$

$$\frac{y{:}F_f\ A,\ z{:}F_f\ B\ [y\otimes z] \vdash F_f\ (A \times B)}{x{:}\ F_f\ A \otimes F_f\ B\ [x] \vdash F_f\ (A \times B)}$$

$A\otimes B := F_{y\otimes z}(y{:}A, z{:}B) \qquad A\times B := F_{y\times z}(y{:}A, z{:}B)$

$$\cfrac{\cfrac{y{:}A,\ z{:}F_f\ B\ [f(y)\otimes z] \vdash F_f\ (A \times B)}{y{:}F_f\ A,\ z{:}F_f\ B\ [y \otimes z] \vdash F_f\ (A \times B)}}{x{:}\ F_f\ A \otimes F_f\ B\ [x] \vdash F_f\ (A \times B)}$$

$A \otimes B := F_{y \otimes z}(y{:}A, z{:}B)$ $\quad$ $A \times B := F_{y \times z}(y{:}A, z{:}B)$

$$\dfrac{\dfrac{\dfrac{y{:}A,\ z{:}B\ [f(y)\otimes f(z)] \vdash F_f\,(A \times B)}{y{:}A,\ z{:}F_f\,B\ [f(y)\otimes z] \vdash F_f\,(A \times B)}}{y{:}F_f\,A,\ z{:}F_f\,B\ [y\otimes z] \vdash F_f\,(A \times B)}}{x{:}\ F_f\,A \otimes F_f\,B\ [x] \vdash F_f\,(A \times B)}$$

$A\otimes B := F_{y\otimes z}(y{:}A,z{:}B)$ $\quad$ $A\times B := F_{y\times z}(y{:}A,z{:}B)$

$f(y)\otimes f(z)\Rightarrow f(?)$

$$\dfrac{\dfrac{\dfrac{\dfrac{}{y{:}A,\ z{:}B\ [f(y)\otimes f(z)]\vdash F_f\ (A\times B)}}{y{:}A,\ z{:}F_f\ B\ [f(y)\otimes z]\vdash F_f\ (A\times B)}}{y{:}F_f\ A,\ z{:}F_f\ B\ [y\otimes z]\vdash F_f\ (A\times B)}}{x{:}\ F_f\ A\otimes F_f\ B\ [x]\vdash F_f\ (A\times B)}$$

$A\otimes B := F_{y\otimes z}(y{:}A,z{:}B)$ $\quad$ $A\times B := F_{y\times z}(y{:}A,z{:}B)$

$$\frac{\overline{\qquad y{:}A,\ z{:}B\ [f(y)\otimes f(z)] \vdash F_f\,(A \times B) \qquad}}{\dfrac{y{:}A,\ z{:}F_f\,B\ [f(y)\otimes z] \vdash F_f\,(A \times B)}{\dfrac{y{:}F_f\,A,\ z{:}F_f\,B\ [y\otimes z] \vdash F_f\,(A \times B)}{x{:}\ F_f\,A \otimes F_f\,B\ [x] \vdash F_f\,(A \times B)}}}$$

$A\otimes B := F_{y\otimes z}(y{:}A,z{:}B)$ $\quad$ $A\times B := F_{y\times z}(y{:}A,z{:}B)$

$$f(y)\otimes f(z)\Rightarrow f(y\times z)$$

$$\frac{\dfrac{\dfrac{y{:}A,\ z{:}B\ [f(y)\otimes f(z)] \vdash F_f\ (A \times B)}{y{:}A,\ z{:}F_f\ B\ [f(y)\otimes z] \vdash F_f\ (A \times B)}}{y{:}F_f\ A,\ z{:}F_f\ B\ [y\otimes z] \vdash F_f\ (A \times B)}}{x{:}\ F_f\ A \otimes F_f\ B\ [x] \vdash F_f\ (A \times B)}$$

$$A\otimes B := F_{y\otimes z}(y{:}A,z{:}B) \qquad A\times B := F_{y\times z}(y{:}A,z{:}B)$$

$$\dfrac{\dfrac{\dfrac{f(y)\otimes f(z)\Rightarrow f(y\times z) \qquad \dfrac{y{:}A[y]\vdash A \qquad z{:}B[z]\vdash B}{y{:}A,z{:}B[y\times z] \vdash A \times B}}{y{:}A,\ z{:}B\ [f(y)\otimes f(z)]\vdash F_f\ (A \times B)}}{y{:}A,\ z{:}F_f\ B\ [f(y)\otimes z]\vdash F_f\ (A \times B)}}{\dfrac{y{:}F_f\ A,\ z{:}F_f\ B\ [y\otimes z]\vdash F_f\ (A \times B)}{x{:}\ F_f\ A \otimes F_f\ B\ [x]\vdash F_f\ (A \times B)}}$$

$$A\otimes B := F_{y\otimes z}(y{:}A,z{:}B) \qquad A\times B := F_{y\times z}(y{:}A,z{:}B)$$

Today

Develop a logic in which

$\otimes$! $\wedge_*$ $\Box$ F are all instances of one connective

$\multimap$ $\rightarrow$ $-_*$ U are all instances of another

Why?

- Pattern → abstraction
- Tool for studying new situations

Spaces

$\int \dashv \Delta \dashv \Gamma \dashv \nabla$

Sets

Right Adjoints

$$\frac{\Gamma, A \vdash B}{\Gamma \vdash A \multimap B}$$

$$\frac{\Delta \vdash A \quad \Gamma[B] \vdash C}{\Gamma[A \multimap B, \Delta] \vdash C}$$

$$\frac{\Gamma; \cdot \vdash_{\ell} A}{\Gamma \vdash_{c} U\,A}$$

Right Adjoints

$$\frac{\Gamma, A \vdash B}{\Gamma \vdash A \multimap B} \qquad \frac{\Delta \vdash A \quad \Gamma[B] \vdash C}{\Gamma[A \multimap B, \Delta] \vdash C}$$

$$\frac{\Gamma; \cdot \vdash_{\ell} A}{\Gamma \vdash_{c} U\,A} \qquad \frac{\Gamma; \Delta, A \vdash_{\ell} B}{\Gamma, U\,A; \Delta \vdash_{\ell} B}$$

Right Adjoints

$$\frac{\phi, c{:}q \vdash \alpha : p \qquad \Delta\ \mathsf{ctx}_{\phi} \qquad A\ \mathsf{type}_{p}}{U_{\alpha}(\Delta \mid A)\ \mathsf{type}_{q}}$$

Right Adjoints

$$\frac{\phi, c{:}q \vdash \alpha : p \qquad \Delta\ \mathrm{ctx}_{\phi} \qquad A\ \mathrm{type}_p}{U_{\alpha}(\Delta|A)\ \mathrm{type}_q}$$

$$A \multimap B := U_{c \otimes y}(y{:}A|B)$$

Right Adjoints

$$\frac{\phi, c{:}q \vdash \alpha : p \qquad \Delta\ \mathrm{ctx}_{\phi} \qquad A\ \mathrm{type}_{p}}{U_{\alpha}(\Delta|A)\ \mathrm{type}_{q}}$$

Right Adjoints

$$\frac{\phi, c{:}q \vdash \alpha : p \qquad \Delta\ \mathrm{ctx}_{\phi} \qquad A\ \mathrm{type}_{p}}{U_{\alpha}(\Delta | A)\ \mathrm{type}_{q}}$$

$$\frac{\Gamma, \Delta\ [\alpha[\beta/c]] \vdash A}{\Gamma\ [\beta] \vdash U_{\alpha}(\Delta | A)}$$

Right Adjoints

$\Delta\ \mathsf{ctx}_\phi$

$A\ \mathsf{type}_p$

$\Gamma\ \mathsf{ctx}_\psi$

$C\ \mathsf{type}_r$

$$\frac{\beta \Rightarrow \gamma[\alpha[\delta]/y] \qquad \begin{array}{c}\Gamma\ [\delta] \vdash \Delta \\ \Gamma, x{:}A\ [\gamma] \vdash C\end{array}}{\Gamma, x{:}U_\alpha(\Delta \mid A)\ [\beta] \vdash C}$$

$$\psi, x{:}q \vdash \beta : r$$

$$\psi \vdash \delta : \phi \qquad \phi, x{:}q \vdash \alpha : p \qquad \psi, y{:}p \vdash \gamma : r$$

Right Adjoints

$$x{:}X, a{:}A\ [x \otimes a] \vdash Y$$

Right Adjoints

$$\frac{x{:}X, a{:}A\ [x \otimes a] \vdash Y}{x{:}X\ [x] \vdash U_{c \otimes a}(a{:}A \mid Y)}$$

Right Adjoints

$$\dfrac{\dfrac{z{:}F_{x\otimes a}(x{:}X, a{:}A)\ [z] \vdash Y}{x{:}X, a{:}A\ [x \otimes a] \vdash Y}}{x{:}X\ [x] \vdash U_{c\,\otimes\,a}(a{:}A|Y)}$$

Right Adjoints

$$\frac{\dfrac{z{:}F_{x\otimes a}(x{:}X, a{:}A)\ [z] \vdash Y}{x{:}X, a{:}A\ [x \otimes a] \vdash Y}}{x{:}X\ [x] \vdash U_{c\,\otimes\,a}(a{:}A|Y)}$$

$$A \multimap Y$$

Right Adjoints

$$X \otimes A$$

$$\begin{array}{c}
z{:}F_{x \otimes a}(x{:}X, a{:}A)\ [z] \vdash Y \\
\hline\hline
x{:}X, a{:}A\ [x \otimes a] \vdash Y \\
\hline\hline
x{:}X\ [x] \vdash U_{c\, \otimes\, a}(a{:}A|Y)
\end{array}$$

$$A \multimap Y$$

Categorical Semantics

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- $\Gamma\ [\beta] \vdash A$ is a cartesian multicategory over a cartesian 2-multicategory

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- $\Gamma\ [\beta] \vdash A$ is a cartesian multicategory over a cartesian 2-multicategory
- $F_\alpha\,\Delta$ makes this into an opfibration
- $U_\alpha(\Delta|A)$ combines right adjoints for unary F's with closed structure (a *closed fibration)*

Today

Develop a logic in which

$\otimes$! $\wedge$ $*$ $\Box$ F are all instances of one connective

$\multimap$ $\rightarrow$ -$*$ U are all instances of another

Why?

- It's satisfying
- Tool for studying new situations

$$\begin{tikzcd}[column sep=large, row sep=huge] \mathrm{Spaces} \arrow[d, bend right=70, "\int"'] \arrow[d, bend right=30, leftarrow, "\Delta"] \arrow[d, "\Gamma"] \arrow[d, bend left=50, leftarrow, "\nabla"] \\ \mathrm{Sets} \end{tikzcd}$$

$$\int \dashv \Delta \dashv \Gamma \dashv \nabla$$