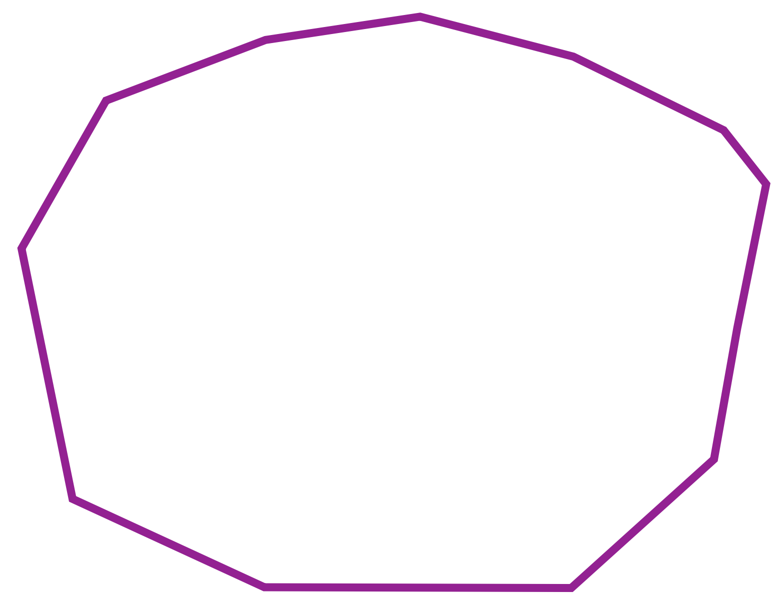
# Adjoint logic with a 2-category of modes

**Dan Licata**
Wesleyan University

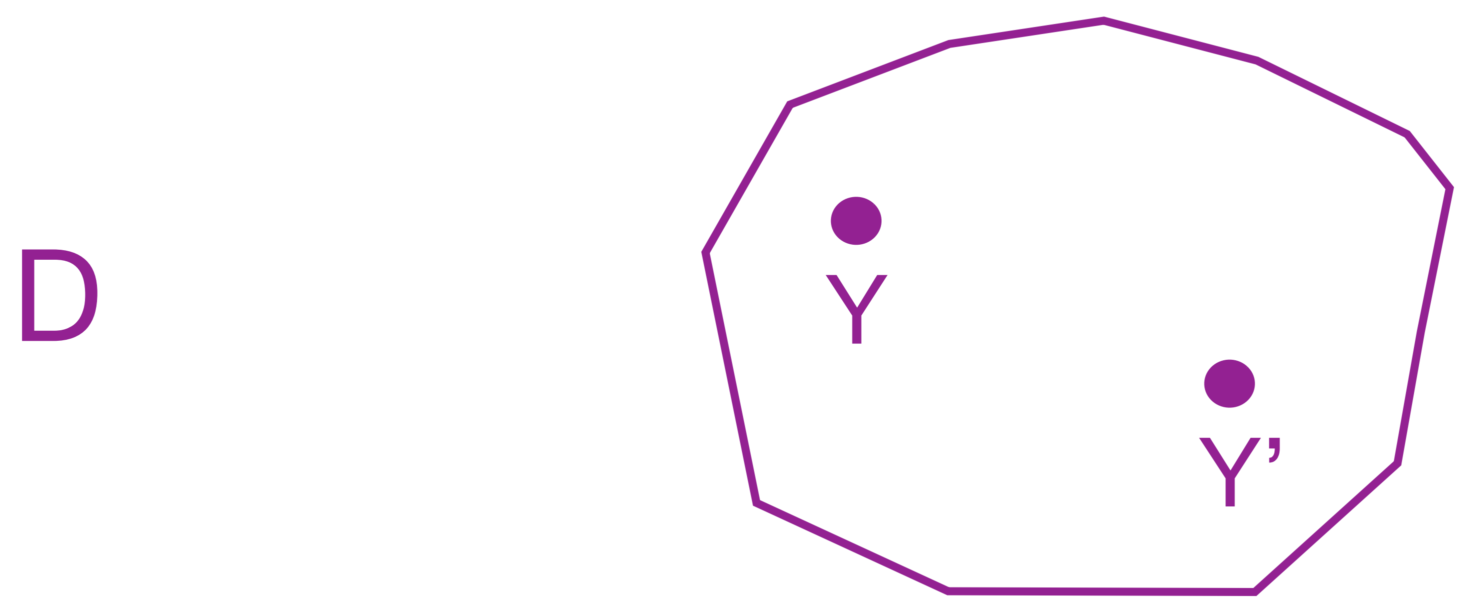
Michael Shulman
University of San Diego

# Adjunctions

D

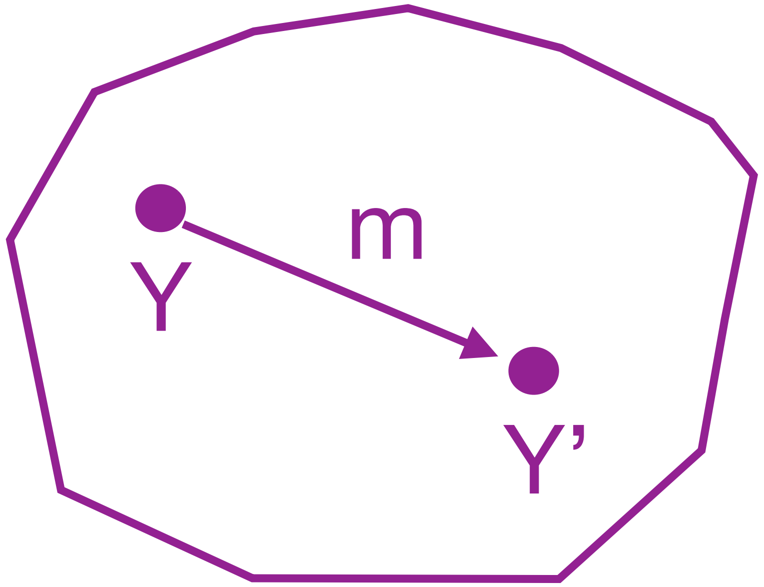


# Adjunctions



# Adjunctions

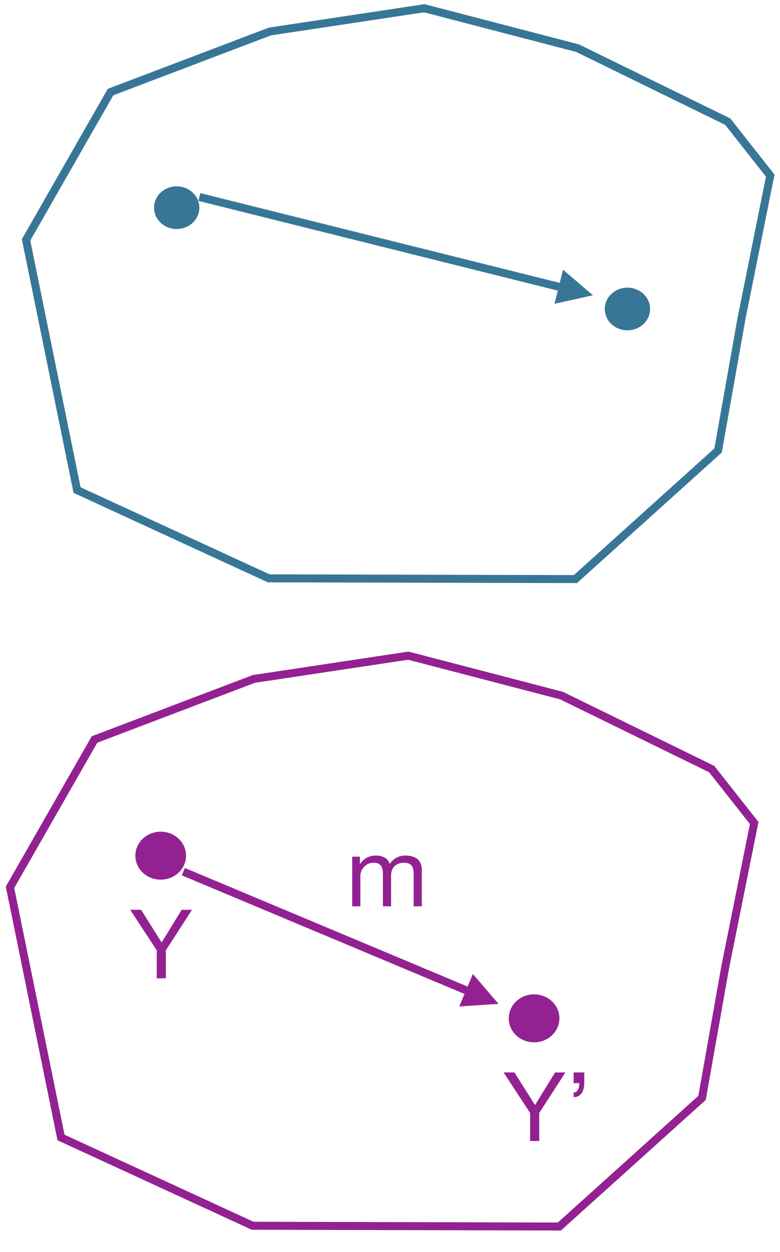
D



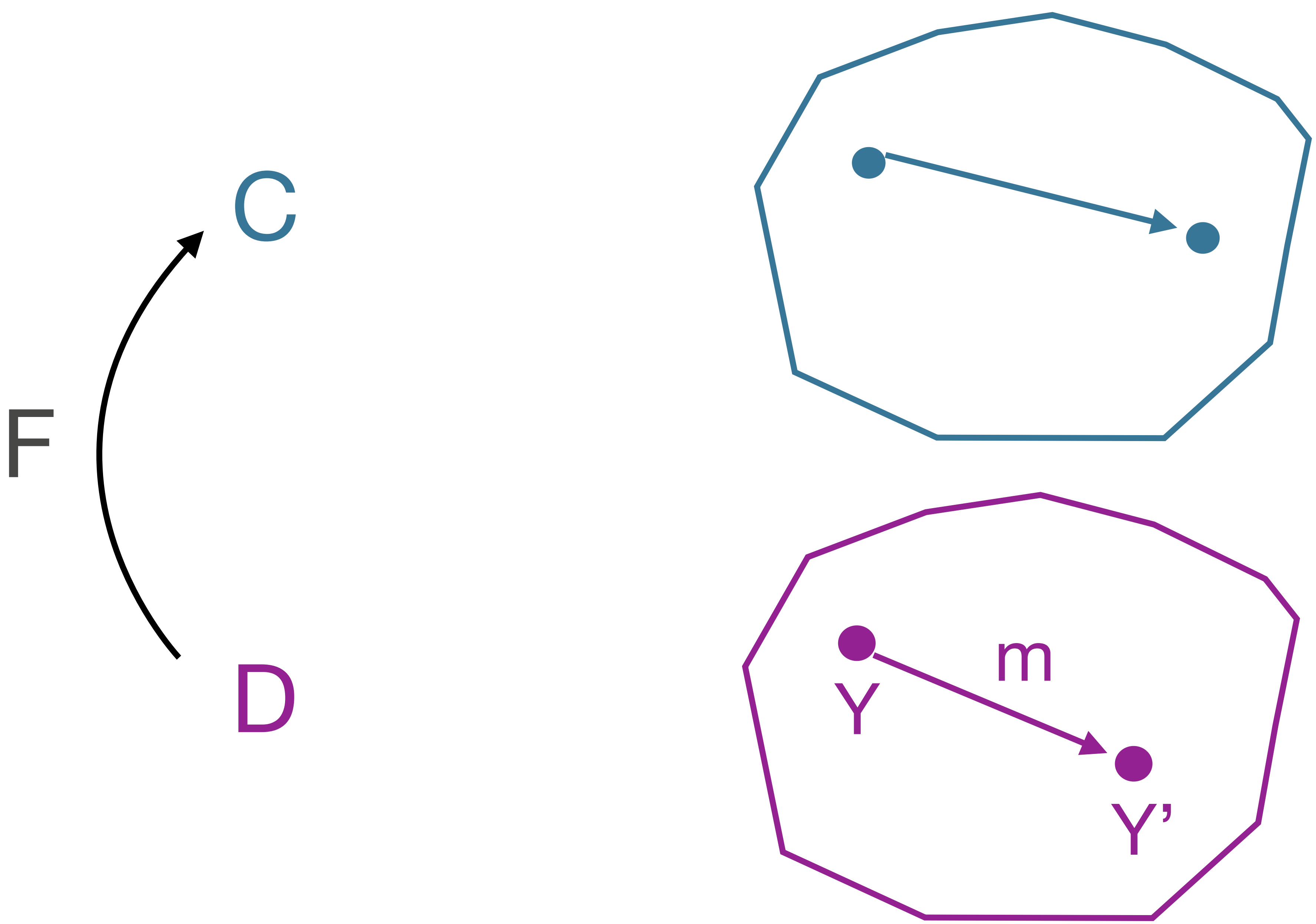
# Adjunctions

C

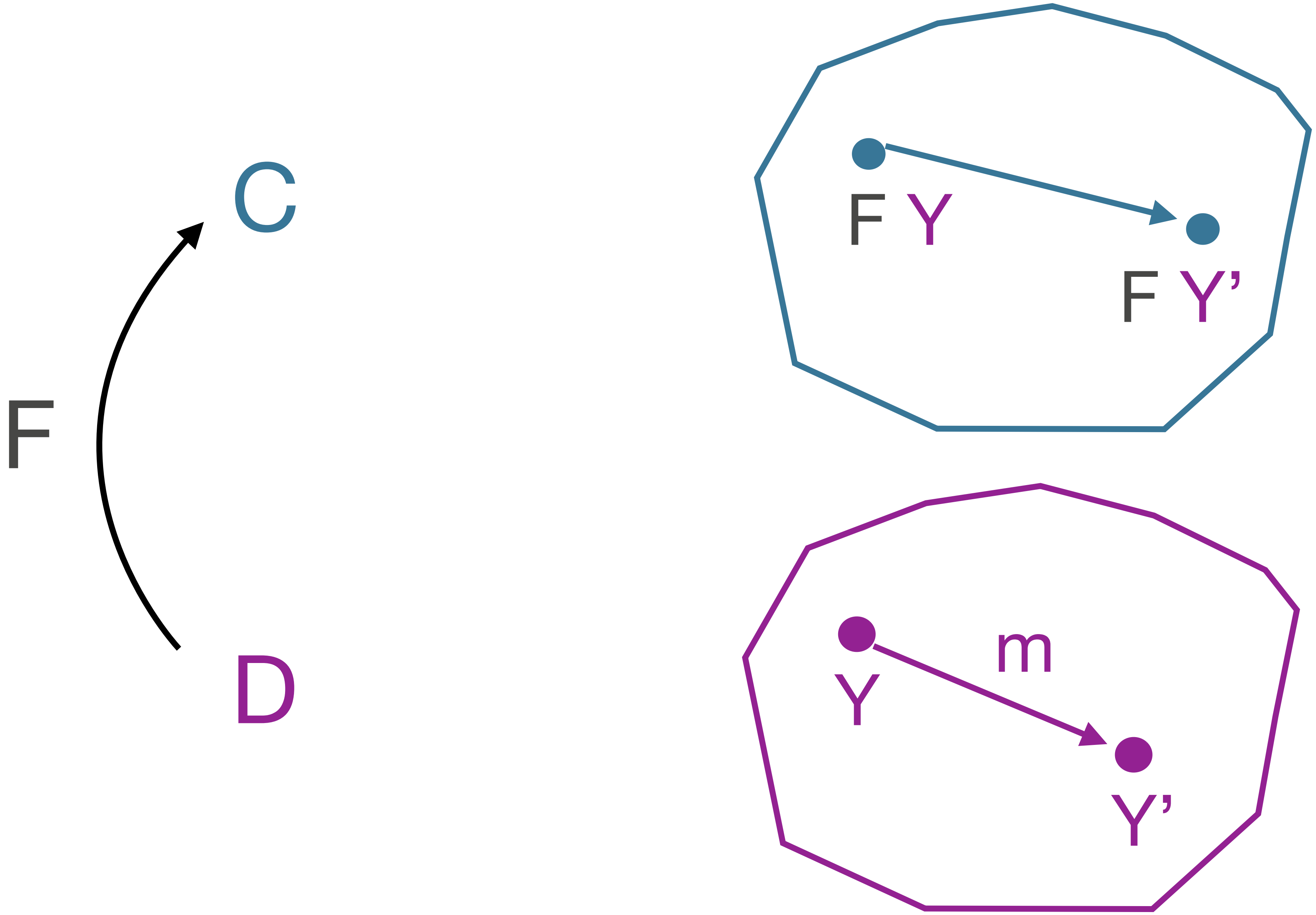
D



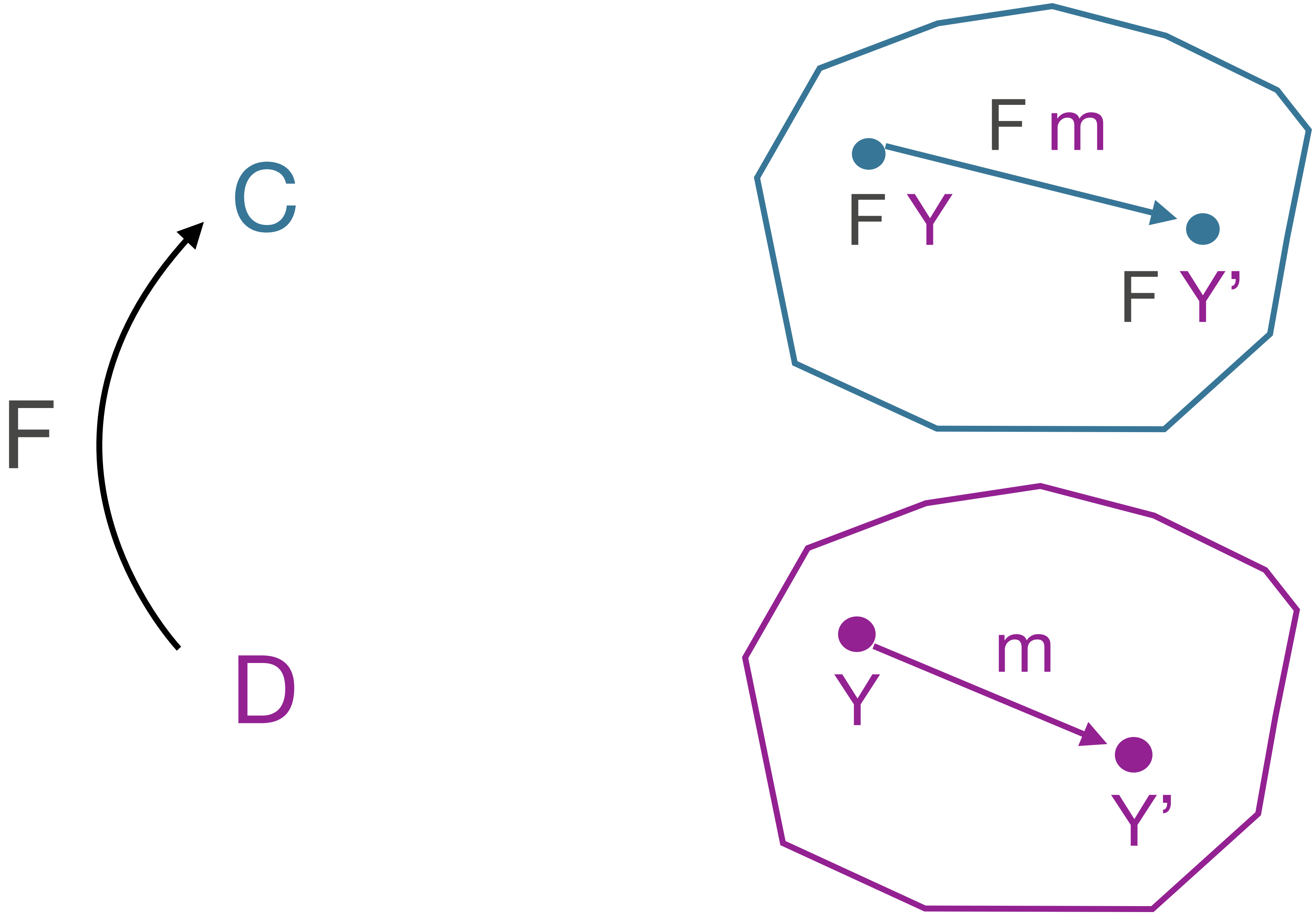
# Adjunctions



# Adjunctions



# Adjunctions



# Adjunctions

$$\begin{tikzcd} C \arrow[d, bend left=60, "U"] \arrow[d, phantom, "\dashv"] \\ D \arrow[u, bend left=60, "F"] \end{tikzcd}$$

$$\frac{F\,Y \to_{C} X}{Y \to_{D} U\,X}$$

# Adjunctions

objects are proposition,
morphisms are proofs

$$
\begin{tikzcd}
L \arrow[d, bend left, "A \supset -"] \\
L \arrow[u, bend left, "- \wedge A"] \arrow[u, phantom, "\dashv"]
\end{tikzcd}
$$

$$
\dfrac{\Gamma \wedge A \vdash B}{\Gamma \vdash A \supset B}
$$

# Adjunctions

$$
\begin{tikzcd}
\mathrm{Groups} \arrow[d, bend left=50, "\mathrm{Underlying}"] \arrow[d, phantom, "\dashv"] \\
\mathrm{Sets} \arrow[u, bend left=50, "\mathrm{Free}"]
\end{tikzcd}
$$

$$
\begin{array}{c}
F\ X \to_{\mathrm{Groups}} Y \\
\hline\hline
X \to_{\mathrm{Sets}} U\ Y
\end{array}
$$

# Proof theory for adjunctions

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- Connections to modal/linear logic:
  - $\Box A := FU\,A$ is a comonad/necessitation
  - $\bigcirc A := UF\,A$ is a monad/possibility

# Proof theory for adjunctions

- Connections to modal/linear logic:
  $\Box$ A := FU A is a comonad/necessitation
  $\bigcirc$ A := UF A is a monad/possibility

- Synthetic mathematics: use logic and type theory to describe categories of interest; formalize math in proof assistants

# Cohesive HoTT [Schreiber,Shulman]

synthetic homotopy theory
as in homotopy type theory

**use types in MLTT to describe homotopy types**

# Cohesive HoTT [Schreiber,Shulman]

synthetic homotopy theory
as in homotopy type theory
+
synthetic topology
as in *axiomatic cohesion*

**use types in MLTT to describe homotopy types**

**can also talk about topological or differentiable structure**

# Cohesive HoTT [Schreiber,Shulman]

synthetic homotopy theory
as in homotopy type theory
+
synthetic topology
as in *axiomatic cohesion*

**use types in MLTT to describe homotopy types**

**can also talk about topological or differentiable structure**

- relate "native" HoTT circle to $\{(x,y) \in R^2 \mid x^2 + y^2 = 1\}$
- theoretical physics

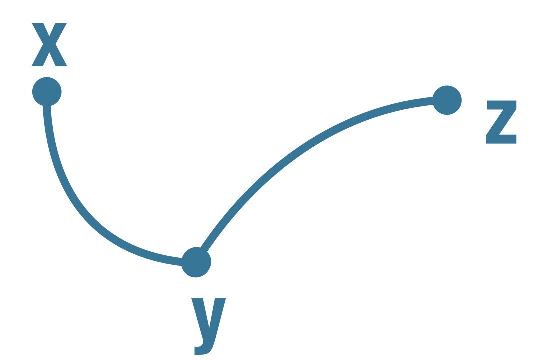
# Axiomatic cohesion [Lawvere]

Sets

{x,y,z}

# Axiomatic cohesion [Lawvere]

Spaces

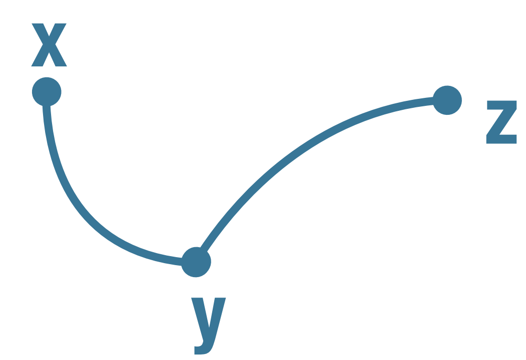



Sets

{x,y,z}

# Axiomatic cohesion [Lawvere]

$$\begin{tikzcd} \text{Spaces} \arrow[d, "\Gamma"] \\ \text{Sets} \end{tikzcd}$$




{x,y,z}

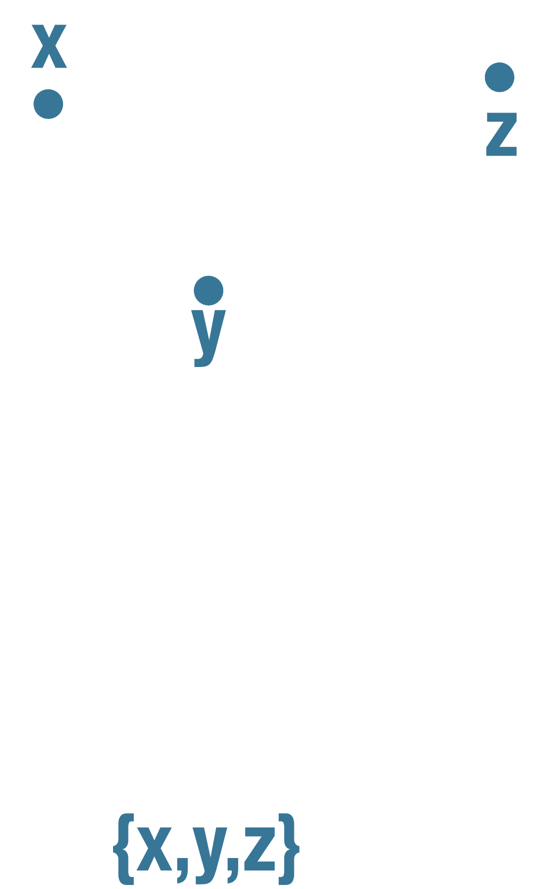
# Axiomatic cohesion [Lawvere]

$$\begin{tikzcd} \text{Spaces} \arrow[d, "\Gamma"] \\ \text{Sets} \end{tikzcd}$$

**{x,y,z}**

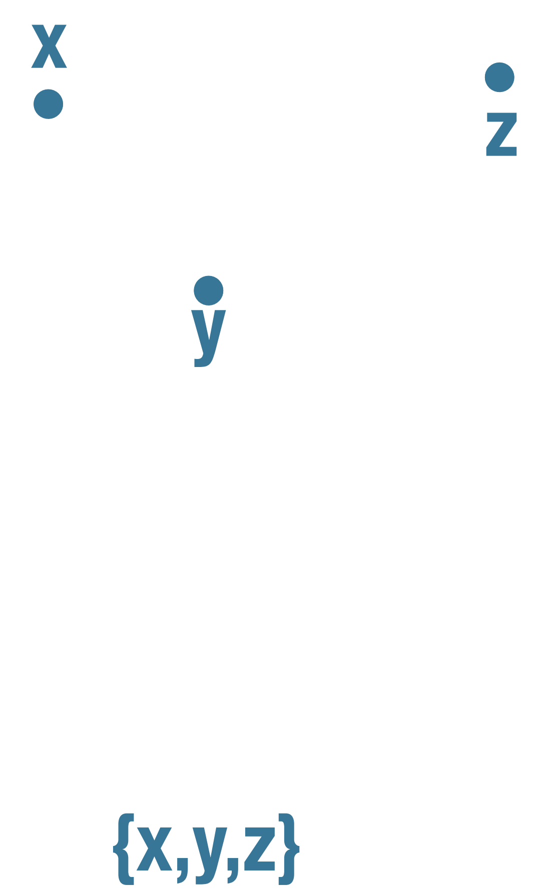
# Axiomatic cohesion [Lawvere]

$$\begin{tikzcd} \text{Spaces} \arrow[d, "\Gamma"] \\ \text{Sets} \arrow[u, bend left, "\Delta"] \end{tikzcd}$$



# Axiomatic cohesion [Lawvere]

$$
\begin{tikzcd}
\text{Spaces} \arrow[d, "\Gamma"] \\
\text{Sets} \arrow[u, bend left, "\Delta"] \arrow[u, phantom, "\dashv"]
\end{tikzcd}
$$




$$
\frac{\Delta X \to_{\text{Spaces}} S}{X \to_{\text{Sets}} \Gamma S}
$$

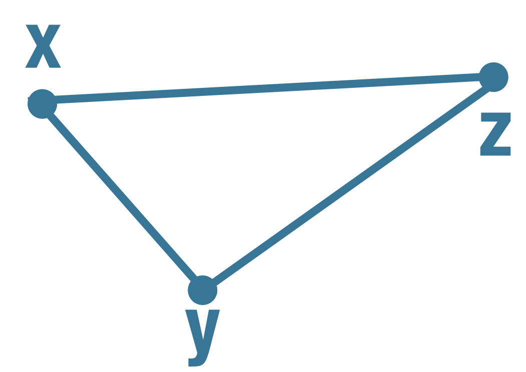
# Axiomatic cohesion [Lawvere]

$$\begin{tikzcd} \text{Spaces} \arrow[d, "\Gamma"] \\ \text{Sets} \end{tikzcd}$$

{x,y,z}

# Axiomatic cohesion [Lawvere]

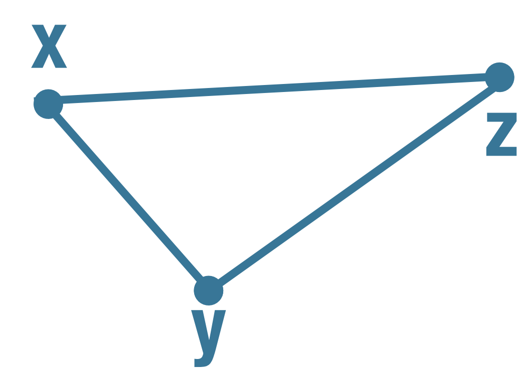
$$\begin{tikzcd}
\text{Spaces} \arrow[d, "\Gamma"] \\
\text{Sets} \arrow[u, bend right, "\nabla"']
\end{tikzcd}$$




{x,y,z}

# Axiomatic cohesion [Lawvere]

$$
\begin{tikzcd}
\text{Spaces} \arrow[d, shift right=4, "\Gamma"'] \arrow[d, phantom, "\Gamma \dashv \nabla"] \\
\text{Sets} \arrow[u, bend right=50, "\nabla"']
\end{tikzcd}
$$

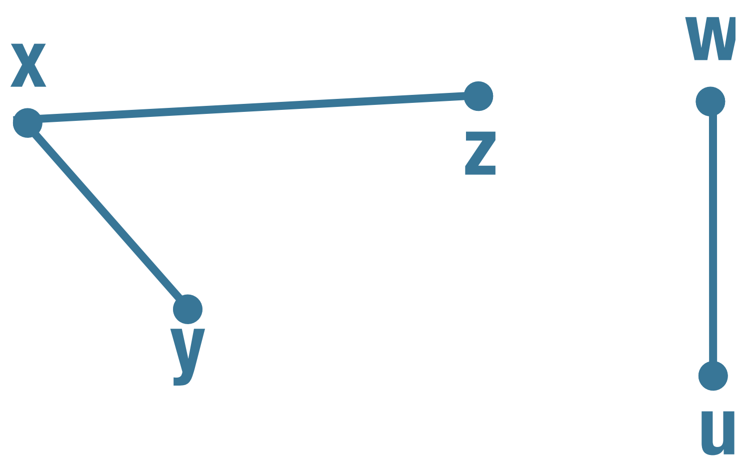



{x,y,z}

$$
\dfrac{S \to_{\text{Spaces}} \nabla\, Y}{\Gamma\, S \to_{\text{Sets}} Y}
$$

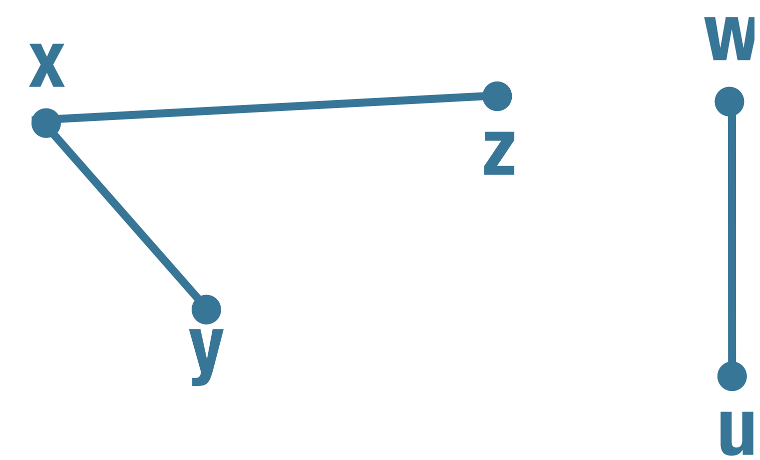
# Axiomatic cohesion [Lawvere]

$$
\begin{tikzcd}[row sep=huge]
\text{Spaces} \arrow[d, "\Gamma"] \\
\text{Sets} \arrow[u, bend left=60, "\Delta"] \arrow[u, bend right=60, "\nabla"']
\end{tikzcd}
\qquad \Delta \dashv \Gamma \dashv \nabla
$$



# Axiomatic cohesion [Lawvere]

$$\begin{tikzcd}[column sep=large] \mathrm{Spaces} \arrow[d, bend right=60, "\Pi_0"'] \arrow[d, "\Gamma"] \\ \mathrm{Sets} \arrow[u, bend left=30, "\Delta"] \arrow[u, bend right=40, "\nabla"'] \end{tikzcd} \qquad \Pi_0 \dashv \Delta \dashv \Gamma \dashv \nabla$$

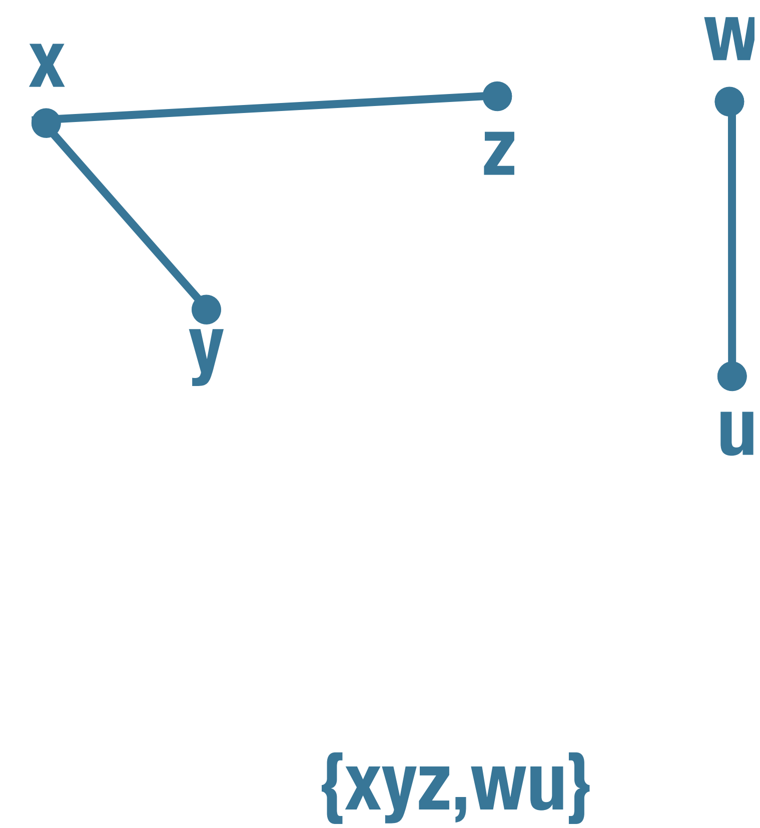



{xyz,wu}

# Axiomatic cohesion [Lawvere]

$$\begin{tikzcd}[column sep=large]
\mathrm{Spaces} \arrow[d, bend right=70, "\Pi_0"'] \arrow[d, "\Gamma"] \\
\mathrm{Sets} \arrow[u, bend left=40, "\Delta"] \arrow[u, bend right=70, "\nabla"']
\end{tikzcd}$$

$$\Pi_0 \dashv \Delta \dashv \Gamma \dashv \nabla$$




$$\frac{S \to_{\mathrm{Spaces}} \Delta Y}{\Pi_0 S \to_{\mathrm{Sets}} Y}$$

# Axiomatic cohesion [Lawvere]

$$\begin{tikzcd}[column sep=large, row sep=huge]
\text{“Spaces”} \arrow[d, bend right=80, "\Pi_0"'] \arrow[d, "\Gamma"] \\
\text{“Sets”} \arrow[u, bend left=40, "\Delta"] \arrow[u, bend right=80, "\nabla"']
\end{tikzcd}
\qquad \Pi_0 \dashv \Delta \dashv \Gamma \dashv \nabla$$

**Abstraction/interface** for:

- topology
- smooth/differentiable
- …

# Type theory **has** A ∧ B, A ⊃ B, Bool, etc.

**Need** logical connectives for

$$\begin{tikzcd}[column sep=huge, row sep=huge] \text{Spaces} \arrow[d, bend right=80, "\Pi_0"'] \arrow[d, "\Gamma"] \\ \text{Sets} \arrow[u, bend left=40, "\Delta"] \arrow[u, bend right=80, "\nabla"'] \end{tikzcd}$$

$$\Pi_0 \dashv \Delta \dashv \Gamma \dashv \nabla$$

# Proof theory of adjunctions

- Benton '94, Benton and Wadler '96:
  adjunction between linear and structural logics,
  decompose ! as FU and ○ as UF

- Reed '09: generalize to a **preorder** of modes,
  decompose Pfenning-Davies □, ○, ◇,
  c.f. multimodal logics and subexponentials

## Mode signature (preorder)

## Modes of props + Connectives

## Mode signature (preorder)

## Modes of props + Connectives

p

p

## Mode signature (preorder)

## Modes of props + Connectives

p

p

q

q

r

r

## Mode signature (preorder)

$$\begin{tikzcd} p \arrow[d] \\ q \\ r \end{tikzcd}$$

## Modes of props + Connectives

p

q

r

## Mode signature (preorder)

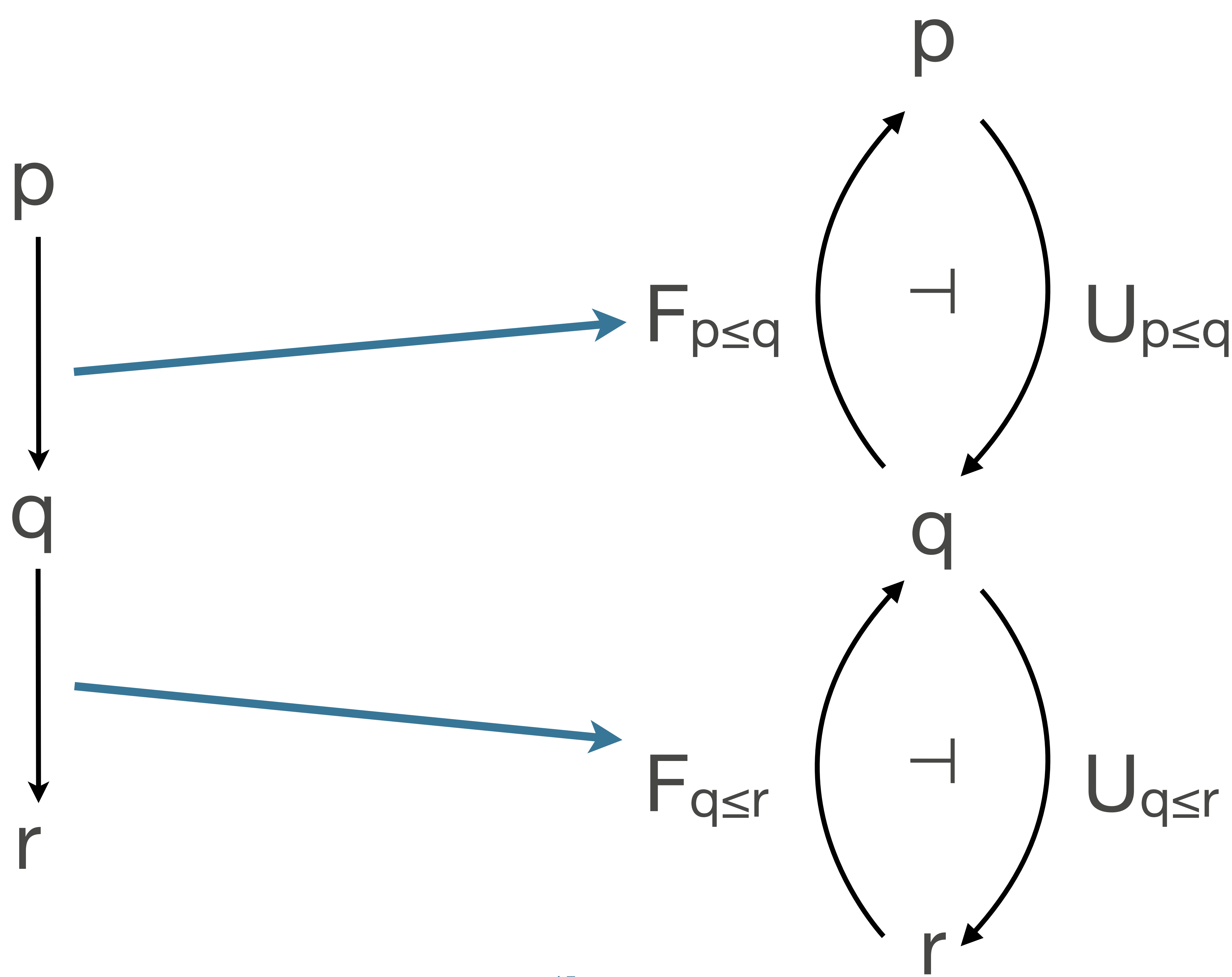
$$
\begin{tikzcd}
p \arrow[d] \\
q \\
\\
r
\end{tikzcd}
$$

## Modes of props + Connectives

$$
\begin{tikzcd}
p \arrow[d, bend left=50, "U_{p \leq q}"] \arrow[d, phantom, "\dashv"] \\
q \arrow[u, bend left=50, "F_{p \leq q}"] \\
\\
r
\end{tikzcd}
$$

**Mode signature (preorder)**

**Modes of props + Connectives**



## Mode signature (preorder)

## Modes of props + Connectives

$$\begin{tikzcd}
p \arrow[d] \\
q \arrow[d] \\
r
\end{tikzcd}
\qquad\qquad
\begin{tikzcd}[row sep=huge]
p \arrow[d, bend left, "U_{p\leq q}"] \arrow[dd, bend left=80, "\cong U_{p\leq r}"] \\
q \arrow[u, bend left, "F_{p\leq q}"] \arrow[u, phantom, "\dashv"] \arrow[d, bend left, "U_{q\leq r}"] \\
r \arrow[u, bend left, "F_{q\leq r}"] \arrow[u, phantom, "\dashv"] \arrow[uu, bend left=80, "F_{p\leq r}\cong"]
\end{tikzcd}$$

$$\begin{tikzcd}
\text{Spaces} \arrow[d, bend right=70, "\Pi_0"'] \arrow[d, "\Gamma"] \\
\text{Sets} \arrow[u, bend left=40, "\Delta"] \arrow[u, bend right=70, "\nabla"']
\end{tikzcd}$$

$$\Pi_0 \dashv \Delta \dashv \Gamma \dashv \nabla$$

$$\begin{tikzcd}[column sep=huge, row sep=huge]
\text{Spaces} \arrow[d, bend right=80, "\Pi_0"'] \arrow[d, "\Gamma"] \\
\text{Sets} \arrow[u, bend left=40, "\Delta"] \arrow[u, bend right=40, "\nabla"']
\end{tikzcd}$$

Need **different** adjunctions
between same categories

## Mode 1-category

$p$

$q$

## Connectives

$p$

$q$

## Mode 1-category

$$\begin{tikzcd} p \arrow[d, bend right, "\alpha"'] \\ q \end{tikzcd}$$

## Connectives

$$\begin{tikzcd} p \\ q \end{tikzcd}$$

## Mode 1-category

$$\begin{tikzcd} p \arrow[d, bend right, "\alpha"'] \\ q \end{tikzcd}$$

## Connectives

$$\begin{tikzcd} p \arrow[d, bend left, "U_\alpha"] \\ q \arrow[u, bend left, "F_\alpha"] \arrow[u, phantom, "\dashv"] \end{tikzcd}$$

**Mode 1-category**

$$\begin{tikzcd}
p \arrow[d, bend right=50, "\alpha"'] \arrow[d, bend left=50, "\beta"] \\
q
\end{tikzcd}$$

**Connectives**

$$\begin{tikzcd}
{} \arrow[d, bend left=50, "U_\alpha"] \arrow[d, phantom, "\dashv"] & p \\
{} \arrow[u, bend left=50, "F_\alpha"] & q
\end{tikzcd}$$

## Mode 1-category

$$
\begin{tikzcd}
p \arrow[d, bend right=50, "\alpha"'] \arrow[d, bend left=50, "\beta"] \\
q
\end{tikzcd}
$$

## Connectives

$$
\begin{tikzcd}[column sep=large]
p \arrow[d, bend left=20, "U_\alpha"'] \arrow[d, bend left=70, "U_\beta"] \\
q \arrow[u, bend left=70, "F_\alpha"] \arrow[u, bend left=20, "F_\beta"']
\end{tikzcd}
$$

Spaces

$$\Pi_0 \dashv \Delta \qquad \Gamma \dashv \nabla$$

Sets

$$
\begin{tikzcd}[row sep=huge, column sep=huge]
\mathrm{Spaces} \arrow[d, bend right=60, "\Pi_0"'] \arrow[d, "\Gamma"] \\
\mathrm{Sets} \arrow[u, bend left=40, "\Delta"] \arrow[u, bend right=60, "\nabla"']
\end{tikzcd}
$$

$$\Pi_0 \dashv \Delta \dashv \Gamma \dashv \nabla$$

Need different adjunctions
with **relationships** between them

## Mode 2-category

$$\begin{tikzcd} p \arrow[d, bend right, "\alpha"'] \arrow[d, bend left, "\beta"] \\ q \end{tikzcd}$$

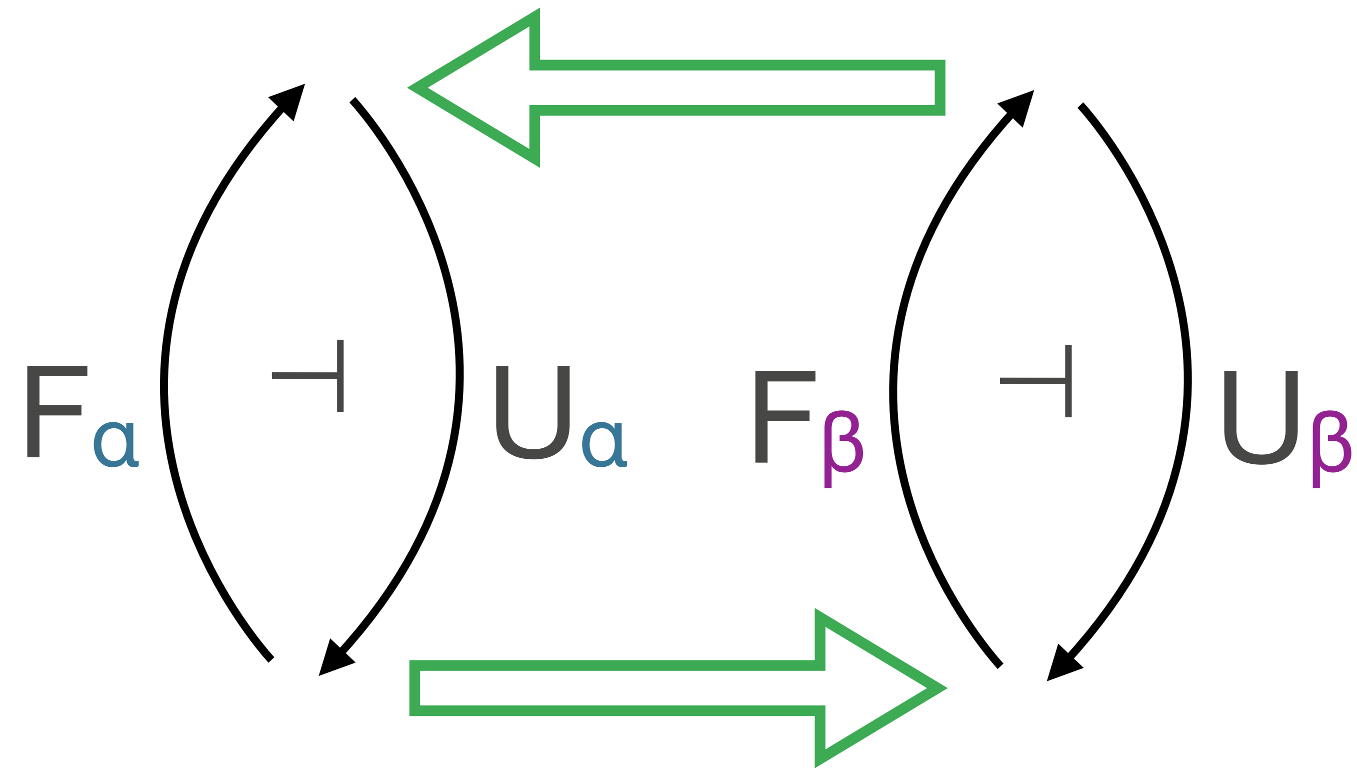
## Connectives + proofs

$$\begin{tikzcd} {} \arrow[d, bend left, "U_\alpha"] \arrow[d, phantom, "\dashv"] \\ {} \arrow[u, bend left, "F_\alpha"] \end{tikzcd} \qquad \begin{tikzcd} {} \arrow[d, bend left, "U_\beta"] \arrow[d, phantom, "\dashv"] \\ {} \arrow[u, bend left, "F_\beta"] \end{tikzcd}$$

## Mode 2-category

$$\begin{tikzcd} p \arrow[d, bend right=60, "\alpha"'] \arrow[d, bend left=60, "\beta"] \arrow[d, phantom, "\overset{e}{\Longrightarrow}"] \\ q \end{tikzcd}$$

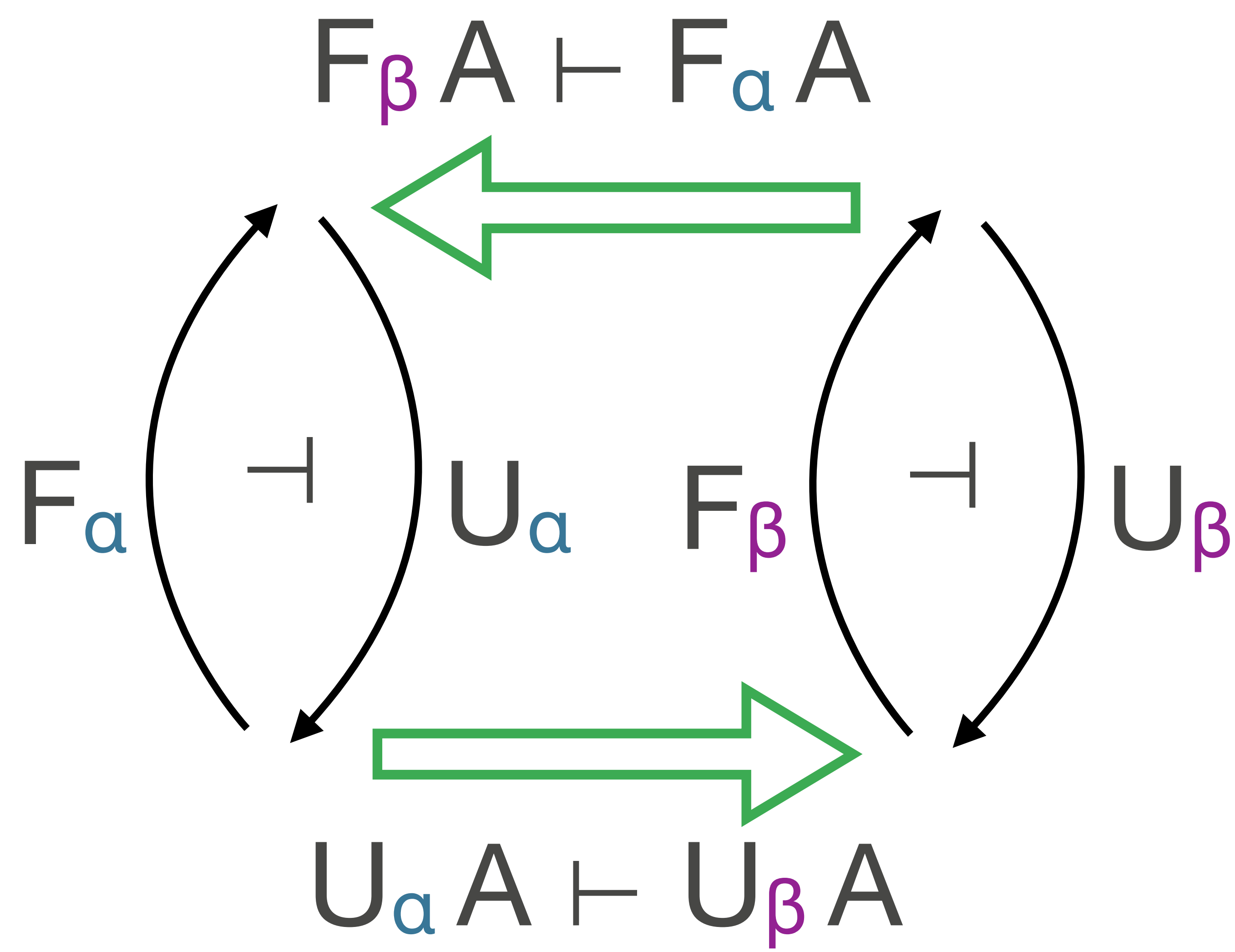
## Connectives + proofs



## Mode 2-category

$$\begin{tikzcd} p \arrow[d, bend right=50, "\alpha"'] \arrow[d, bend left=50, "\beta"] \arrow[d, phantom, "\overset{e}{\Longrightarrow}"] \\ q \end{tikzcd}$$

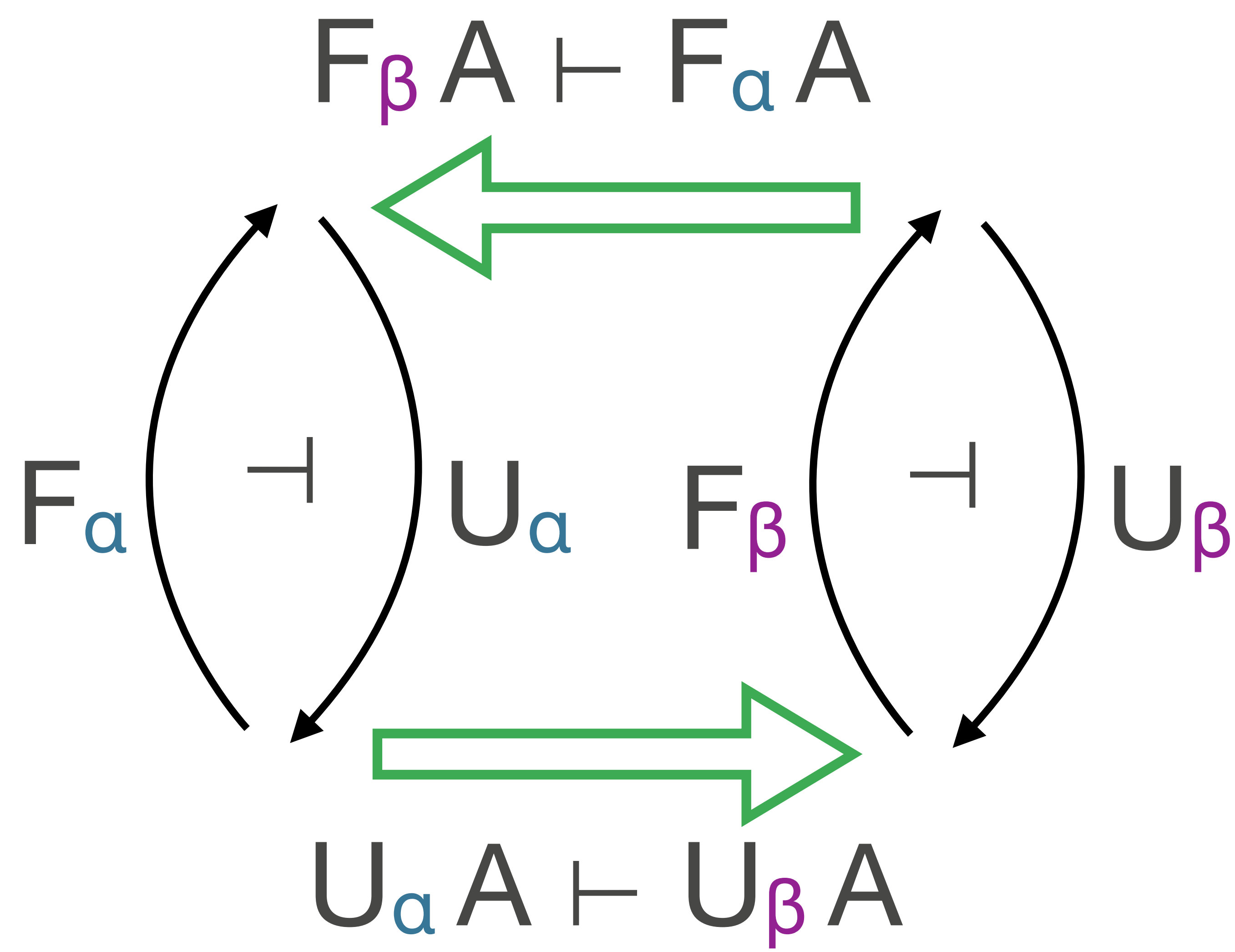
## Connectives + proofs



## Mode 2-category

$$\begin{tikzcd} p \arrow[d, bend right=60, "\alpha"'] \arrow[d, bend left=60, "\beta"] \arrow[d, phantom, "\overset{e}{\Longrightarrow}"] \\ q \end{tikzcd}$$

## Connectives + proofs




## Pseudofunctoriality (1-cells):

$F_1\,A \cong A \cong U_1\,A \qquad F_{\gamma\circ\alpha}\,A \cong F_\alpha\,F_\gamma\,A \qquad U_{\gamma\circ\alpha}\,A \cong U_\gamma\,U_\alpha\,A$

objects c,s

d : s ≧ c

n : c ≧ s

unit : 1 ⟹ n○d

counit : d○n ⟹ 1

+ triangle identities

$$
\begin{tikzcd}
\mathrm{Spaces} \arrow[d, "\Gamma"] \\
\mathrm{Sets} \arrow[u, bend left=60, "\Delta"] \arrow[u, bend right=60, "\nabla"']
\end{tikzcd}
$$

A triple adjunction is an
adjunction of adjunctions

# Naïve calculus

$$\frac{F_\alpha A \vdash B}{A \vdash U_\alpha B} \qquad \frac{A \vdash U_\alpha B}{F_\alpha A \vdash B}$$

# Naïve calculus

$$\frac{F_\alpha A \vdash B}{A \vdash U_\alpha B} \qquad \frac{A \vdash U_\alpha B}{F_\alpha A \vdash B} \qquad \frac{A \vdash B}{F_\alpha A \vdash F_\alpha B} \qquad \frac{A \vdash B}{U_\alpha A \vdash U_\alpha B}$$

# Naïve calculus

$$\frac{F_\alpha A \vdash B}{A \vdash U_\alpha B} \qquad \frac{A \vdash U_\alpha B}{F_\alpha A \vdash B} \qquad \frac{A \vdash B}{F_\alpha A \vdash F_\alpha B} \qquad \frac{A \vdash B}{U_\alpha A \vdash U_\alpha B}$$

$$\frac{}{F_1 A \vdash A} \qquad \frac{}{A \vdash F_1 A} \qquad \frac{}{F_{\gamma\circ\alpha} A \vdash F_\alpha F_\gamma A} \qquad \frac{}{F_\alpha F_\gamma A \vdash F_{\gamma\circ\alpha} A}$$

# Naïve calculus

$$\frac{F_\alpha\, A \vdash B}{A \vdash U_\alpha\, B} \qquad \frac{A \vdash U_\alpha\, B}{F_\alpha\, A \vdash B} \qquad \frac{A \vdash B}{F_\alpha\, A \vdash F_\alpha\, B} \qquad \frac{A \vdash B}{U_\alpha\, A \vdash U_\alpha\, B}$$

$$\frac{}{F_1\, A \vdash A} \qquad \frac{}{A \vdash F_1\, A} \qquad \frac{}{F_{\gamma\circ\alpha}\, A \vdash F_\alpha\, F_\gamma\, A} \qquad \frac{}{F_\alpha\, F_\gamma\, A \vdash F_{\gamma\circ\alpha}\, A}$$

$$\frac{}{U_1\, A \vdash A} \qquad \frac{}{A \vdash U_1\, A} \qquad \frac{}{U_{\gamma\circ\alpha}\, A \vdash U_\gamma\, U_\alpha\, A} \qquad \frac{}{U_\gamma\, U_\alpha\, A \vdash U_{\gamma\circ\alpha}\, A}$$

# Naïve calculus

$$\frac{F_\alpha A \vdash B}{A \vdash U_\alpha B} \qquad \frac{A \vdash U_\alpha B}{F_\alpha A \vdash B} \qquad \frac{A \vdash B}{F_\alpha A \vdash F_\alpha B} \qquad \frac{A \vdash B}{U_\alpha A \vdash U_\alpha B}$$

$$\frac{}{F_1 A \vdash A} \qquad \frac{}{A \vdash F_1 A} \qquad \frac{}{F_{\gamma\circ\alpha} A \vdash F_\alpha F_\gamma A} \qquad \frac{}{F_\alpha F_\gamma A \vdash F_{\gamma\circ\alpha} A}$$

$$\frac{}{U_1 A \vdash A} \qquad \frac{}{A \vdash U_1 A} \qquad \frac{}{U_{\gamma\circ\alpha} A \vdash U_\gamma U_\alpha A} \qquad \frac{}{U_\gamma U_\alpha A \vdash U_{\gamma\circ\alpha} A}$$

$$\frac{\alpha \Longrightarrow \beta}{F_\beta A \vdash F_\alpha A} \qquad \frac{\alpha \Longrightarrow \beta}{U_\alpha A \vdash U_\beta A}$$

# Naïve calculus

$$\frac{F_\alpha A \vdash B}{A \vdash U_\alpha B} \qquad \frac{A \vdash U_\alpha B}{F_\alpha A \vdash B} \qquad \frac{A \vdash B}{F_\alpha A \vdash F_\alpha B} \qquad \frac{A \vdash B}{U_\alpha A \vdash U_\alpha B}$$

$$\frac{\overline{F_1 A \vdash A}}{U_1 A \vdash A} \qquad \frac{\overline{A \vdash F_1 A}}{A \vdash U_1 A} \qquad \frac{\overline{F_{\gamma\circ\alpha} A \vdash F_\alpha F_\gamma A}}{U_{\gamma\circ\alpha} A \vdash U_\gamma U_\alpha A} \qquad \frac{\overline{F_\alpha F_\gamma A \vdash F_{\gamma\circ\alpha} A}}{U_\gamma U_\alpha A \vdash U_{\gamma\circ\alpha} A}$$

$$\frac{\alpha \Longrightarrow \beta}{F_\beta A \vdash F_\alpha A} \qquad \frac{\alpha \Longrightarrow \beta}{U_\alpha A \vdash U_\beta A}$$

$$\frac{}{A \vdash A} \qquad \frac{A \vdash B \quad B \vdash C}{A \vdash C}$$

# Better calculus

- Only left and right rules for $F_\alpha$ and $U_\alpha$
- Cut (composition) and identity admissible
- Subformula property
- Simpler equational theory
- Polarity/focusing story: F positive and U negative
- Still interprets mode theory correctly: everything on last page is provable

# Mixed category entailment

[generalization of Benton,Wadler,Reed]

**morphism in mode category**

$$A\ [\alpha : q \geq p] \vdash B$$

**type with mode q**

**type with mode p**

# Mixed category entailment

[generalization of Benton,Wadler,Reed]

**morphism in mode category**

$$A\ [\alpha : q \geq p] \vdash B$$

**type with mode q** **type with mode p**

means (equivalently) $F_\alpha A \to B$ or $A \to U_\alpha B$

# F Left

$$\frac{A\,[\alpha \circ \beta] \vdash B}{F_{\alpha:r \geq q}\, A\,[\beta : q \geq p] \vdash B}$$

# F Left

$$\frac{A\,[\alpha \circ \beta] \vdash B}{F_{\alpha:r\geq q}\,A\,[\beta : q \geq p] \vdash B}$$

**Meaning of sequent:**

$$\frac{A\,[\alpha] \vdash B}{F_{\alpha}\,A\,[1] \vdash B}$$

# F Left

$$\frac{A\,[\alpha \circ \beta] \vdash B}{F_{\alpha : r \geq q}\, A\,[\beta : q \geq p] \vdash B}$$

**Meaning of sequent:**

$$\frac{A\,[\alpha] \vdash B}{F_{\alpha}\, A\,[1] \vdash B}$$

**Composition:**

$$F_{\alpha \circ \beta}\, A \cong F_{\beta}\, F_{\alpha}\, A$$

# F Left

$F_{\alpha \circ \beta} A$

$$\frac{A\ [\alpha \circ \beta] \vdash B}{F_{\alpha : r \geq q}\ A\ [\beta : q \geq p] \vdash B}$$

**Meaning of sequent:**

$$\frac{A\ [\alpha] \vdash B}{F_{\alpha}\ A\ [1] \vdash B}$$

**Composition:**

$$F_{\alpha \circ \beta}\ A \cong F_{\beta}\ F_{\alpha}\ A$$

# F Left

$F_{\alpha \circ \beta} A$

$$\frac{A\ [\alpha \circ \beta] \vdash B}{F_{\alpha : r \geq q}\ A\ [\beta : q \geq p] \vdash B}$$

$F_\beta F_\alpha A$

**Meaning of sequent:**

$$\frac{A\ [\alpha] \vdash B}{F_\alpha A\ [1] \vdash B}$$

**Composition:**

$$F_{\alpha \circ \beta} A \cong F_\beta F_\alpha A$$

# F Right

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**Functoriality:**

$$\frac{A\ [1] \vdash B}{A\ [\alpha] \vdash F_{\alpha} B}$$

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$$\frac{A\,[1] \vdash B}{A\,[\alpha] \vdash F_\alpha B}$$

**Functoriality + Composition:**

$$\frac{A\,[\gamma] \vdash B}{A\,[\gamma \circ \alpha] \vdash F_\alpha B}$$

# F Right

**Functoriality:**

$$\frac{A\ [1] \vdash B}{A\ [\alpha] \vdash F_\alpha B}$$

**Functoriality + Composition:**

$F_\gamma A$

$$\frac{A\ [\gamma] \vdash B}{A\ [\gamma \circ \alpha] \vdash F_\alpha B}$$

$F_{\gamma \circ \alpha} A \cong F_\alpha F_\gamma A$

# F Right

$F_\gamma A$

**Functoriality:**

$$\frac{A\,[1] \vdash B}{A\,[\alpha] \vdash F_\alpha B}$$

**Functoriality + Composition:**

$$\frac{A\,[\gamma] \vdash B}{A\,[\gamma \circ \alpha] \vdash F_\alpha B}$$

$F_{\gamma\circ\alpha} A \cong F_\alpha F_\gamma A$

**Action of 2-cells:**

$$\frac{e : \alpha \Longrightarrow \beta}{A\,[\beta] \vdash F_\alpha A}$$

# F Right

$$\frac{\gamma{:}r \geq q \qquad e : \gamma \circ \alpha \Longrightarrow \beta \qquad A\ [\gamma] \vdash B}{A\ [\beta{:}r \geq p] \vdash F_{\alpha:q\geq p}\, B}$$

# F Right

$$\frac{\gamma{:}r \geq q \qquad e : \gamma \circ \alpha \Longrightarrow \beta \qquad A\ [\gamma] \vdash B}{A\ [\beta{:}r \geq p] \vdash F_{\alpha : q \geq p}\, B}$$

$$\begin{tikzcd}
q & & r \\
& p \arrow[ul, "\alpha"] \arrow[ur, "\beta"'] &
\end{tikzcd}$$

# F Right

$$\frac{\gamma{:}r \geq q \qquad e : \gamma \circ \alpha \Longrightarrow \beta \qquad A\,[\gamma] \vdash B}{A\,[\beta{:}r \geq p] \vdash F_{\alpha:q \geq p}\, B}$$

$$\begin{tikzcd} q \arrow[rr, dashed, "\gamma"] & & r \\ & p \arrow[ul, "\alpha"] \arrow[ur, "\beta"'] & \end{tikzcd}$$

# F Right

$$\frac{\gamma{:}r\geq q \qquad e : \gamma\circ\alpha \Longrightarrow \beta \qquad A\,[\gamma] \vdash B}{A\,[\beta{:}r\geq p] \vdash F_{\alpha:q\geq p}\,B}$$

$$\begin{tikzcd}
q \arrow[rr, dashed, "\gamma"] & & r \\
& p \arrow[ul, "\alpha"] \arrow[ur, "\beta"'] \arrow[u, phantom, "\overset{e}{\Longrightarrow}"] &
\end{tikzcd}$$

# Rules for U are dual

$$\frac{A_r\,[\alpha \circ \beta] \vdash C_p}{F_{\alpha:r\geq q}\,A_r\,[\beta : q \geq p] \vdash C_p}\ \text{FL} \qquad \frac{\gamma : r \geq q \quad \gamma \circ \alpha \Rightarrow \beta \quad C_r\,[\gamma] \vdash A_q}{C_r\,[\beta : r \geq p] \vdash F_{\alpha:q\geq p}\,A_q}\ \text{FR}$$

$$\frac{C_r\,[\beta \circ \alpha] \vdash A_p}{C_r\,[\beta : r \geq q] \vdash U_{\alpha:q\geq p}\,A_p}\ \text{UR} \qquad \frac{\gamma : q \geq p \quad \alpha \circ \gamma \Rightarrow \beta \quad A_q\,[\gamma] \vdash C_p}{U_{\alpha:r\geq q}\,A_q\,[\beta : r \geq p] \vdash C_p}\ \text{UL}$$

# Admissible rules

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**Identity:**

$$\frac{}{A_p\,[1] \vdash A_p}$$

# Admissible rules

**Identity:**

$$\overline{A_p\,[1] \vdash A_p}$$

**Cut:**

$$\frac{A_r\,[\beta] \vdash B_q \quad B_q\,[\alpha] \vdash C_p}{A_r\,[\beta \circ \alpha] \vdash C_p}$$

# Admissible rules

**Identity:**

$$\frac{}{A_p\,[1] \vdash A_p}$$

**Cut:**

$$\frac{A_r\,[\beta] \vdash B_q \quad B_q\,[\alpha] \vdash C_p}{A_r\,[\beta \circ \alpha] \vdash C_p}$$

**Action of 2-cell:**

$$\frac{\alpha \Rightarrow \beta \quad A\,[\alpha] \vdash C}{A\,[\beta] \vdash C}$$

C

S

c

d : s ≥ c

s

d : s ≥ c

$$\begin{tikzcd} c \arrow[d, bend left=60, "U_d"] \\ s \arrow[u, bend left=60, "F_d"] \arrow[u, phantom, "\dashv"] \end{tikzcd}$$

$d : s \geq c$

$n : c \geq s$

$$\begin{tikzcd} c \arrow[d, bend left=50, "U_d"] \arrow[d, phantom, "\dashv"] \\ s \arrow[u, bend left=50, "F_d"] \end{tikzcd}$$

$d : s \geq c$

$n : c \geq s$

$$
\begin{tikzcd}[row sep=huge]
c \arrow[d, bend right=30, "U_d"] \arrow[d, bend left=30, "F_n"'] \\
s \arrow[u, bend left=80, "F_d"] \arrow[u, bend right=80, "U_n"']
\end{tikzcd}
$$

$d : s \geq c$

$n : c \geq s$

$$
\begin{tikzcd}[row sep=huge, column sep=huge]
c \arrow[d, bend left=40, "U_d"] \arrow[d, bend right=40, "F_n"'] \\
s \arrow[u, bend left=80, "F_d"] \arrow[u, bend right=80, "U_n"']
\end{tikzcd}
\qquad F_d \dashv U_d \cong F_n \dashv U_n
$$

$d : s \geq c$

$n : c \geq s$

$\text{counit} : d \circ n \Longrightarrow 1$

$$\begin{tikzcd}[column sep=huge, row sep=huge]
c \arrow[d, bend left=40, "U_d"] \arrow[d, phantom, shift right=6, "\dashv"] \arrow[d, bend right=80, "F_n"'] \\
s \arrow[u, bend left=40, "F_d"] \arrow[u, phantom, shift left=6, "\dashv"] \arrow[u, bend right=80, "U_n"']
\end{tikzcd}$$

$$F_d \dashv U_d \cong F_n \dashv U_n$$

$\mathsf{d} : \mathsf{s} \geq \mathsf{c}$

$\mathsf{n} : \mathsf{c} \geq \mathsf{s}$

$\text{counit} : \mathsf{d} \circ \mathsf{n} \Longrightarrow 1$ $\longrightarrow$

$$\begin{tikzcd}[row sep=huge, column sep=huge]
\mathsf{c} \arrow[d, bend left=40, "U_{\mathsf{d}}"] \arrow[d, phantom, "\dashv \quad \cong \quad \dashv"] \arrow[d, bend right=40, "F_{\mathsf{n}}"'] \\
\mathsf{s} \arrow[u, bend left=80, "F_{\mathsf{d}}"] \arrow[u, bend right=80, "U_{\mathsf{n}}"']
\end{tikzcd}$$

$$\dfrac{\mathsf{d} : \mathsf{s} \geq \mathsf{c} \qquad \mathsf{counit} : \mathsf{d} \circ \mathsf{n} \Rightarrow 1 \qquad \dfrac{1 : \mathsf{c} \geq \mathsf{c} \quad 1 : \mathsf{d} \Rightarrow \mathsf{d} \quad \dfrac{}{A\,[1] \vdash A}\,\text{ident}}{U_{\mathsf{d}}\,A\,[\mathsf{d}] \vdash A}\,\text{UL}}{U_{\mathsf{d}}\,A\,[1] \vdash F_{\mathsf{n}}\,A}\,\text{FR}$$

$d : s \geq c$

$n : c \geq s$

$\text{counit} : d \circ n \Longrightarrow 1$

$$\begin{tikzcd}[row sep=huge]
c \arrow[d, bend left=30, "U_d"] \arrow[d, bend right=30, "F_n"'] \\
s \arrow[u, bend left=80, "F_d"] \arrow[u, bend right=80, "U_n"']
\end{tikzcd}$$

$$F_d \dashv U_d \cong F_n \dashv U_n$$

$d : s \geq c$

$n : c \geq s$

$\text{counit} : d \circ n \Longrightarrow 1$

$\text{unit} : 1 \Longrightarrow n \circ d$

$$\begin{tikzcd}[row sep=huge]
c \arrow[d, bend left=40, "U_d"] \arrow[d, bend right=40, "F_n"'] \\
s \arrow[u, bend left=80, "F_d"] \arrow[u, bend right=80, "U_n"']
\end{tikzcd}$$

$$F_d \dashv U_d \cong F_n \dashv U_n$$

d : s ≥ c

n : c ≥ s

counit : d∘n ⟹ 1

unit : 1 ⟹ n∘d

$$\begin{tikzcd}[column sep=large] & c \arrow[dd, bend left=30, "U_d"] \arrow[dd, bend right=30, "F_n"'] & \\ F_d \dashv U_d \;\cong\; F_n \dashv U_n & & \\ & s \arrow[uu, bend left=70, "F_d"] \arrow[uu, bend right=70, "U_n"'] & \end{tikzcd}$$

$$\dfrac{\dfrac{\dfrac{\mathsf{unit} : 1 \Rightarrow \mathsf{n} \circ \mathsf{d} \qquad \dfrac{}{A\,[1] \vdash A}\ \mathsf{ident}}{A\,[\mathsf{n} \circ \mathsf{d}] \vdash A}\ -_*(-)}{A\,[\mathsf{n}] \vdash U_{\mathsf{d}}\,A}\ \mathsf{UR}}{F_{\mathsf{n}}\,A\,[1] \vdash U_{\mathsf{d}}\,A}\ \mathsf{FL}$$

$$\begin{tikzcd}[row sep=huge, column sep=huge]
C \arrow[d, bend left=40, "U_d"' near start] \arrow[d, bend right=40, "F_n" near start] \arrow[d, phantom, "\dashv \quad U_d \cong F_n \quad \dashv"] \\
S \arrow[u, bend left=80, "F_d"] \arrow[u, bend right=80, "U_n"']
\end{tikzcd}$$

Explains unusual polarity of middle adjoint (commutes with both negatives and postives)

# What’s in the paper/Agda

- Definitions of cut and identity
- Equational theory for sequent derivations
- Soundness and completeness for pseudofunctors from mode theory to the 2-category of adjunctions
- Constructions for any mode theory (coherence natural isomorphisms, FU/UF is a co/monad)
- Mode theories for triple adjunctions (and with extra properties)

**Theorem 2 (Completeness).** *The syntax of adjoint logic determines a pseudofunctor* $\mathscr{M} \to \mathbf{Adj}$*:*

1. *An object* $p$ *of* $\mathscr{M}$ *is sent to the category whose objects are* $A$ $\mathsf{type}_p$ *and morphisms are derivations of* $A\,[1_p] \vdash B$ *quotiented by* $\approx$*, with identities given by* ident *and composition given by* cut.
2. *For each* $q, p$*, there is a functor from the category of morphisms* $q \geq p$ *to the category of adjoint functors between* $q$ *and* $p$.
   - *Each* $\alpha : q \geq p$ *is sent to* $F_\alpha \dashv U_\alpha$ *in* $\mathbf{Adj}$*—*$F_\alpha$ *and* $U_\alpha$ *are functors and they are adjoint.*
   - *Each 2-cell* $e : \alpha \Rightarrow \beta$ *is sent to a conjugate pair of transformations* $(F(e), U(e)) : (F_\alpha \dashv U_\alpha) \to (F_\beta \dashv U_\beta)$*, and this preserves* $1$ *and* $e_1 \cdot e_2$.
3. $F_1\,A \cong A$ *and* $U_1\,A \cong A$ *naturally in* $A$*, and these are conjugate, so there is an adjunction isomorphism* $P^1$ *between* $F_1 \dashv U_1$ *and the identity adjunction.*
4. $F_{\beta\circ\alpha}\,A \cong F_\alpha\,(F_\beta\,A)$ *and* $U_{\beta\circ\alpha}\,A \cong U_\beta\,(U_\alpha\,A)$ *naturally in* $A$*, and these are conjugate, so there is an adjunction isomorphism* $P^\circ(\alpha, \beta)$ *between* $F_{\beta\circ\alpha} \dashv U_{\beta\circ\alpha}$ *and the composition of the adjunctions* $F_\alpha \dashv U_\alpha$ *and* $F_\beta \dashv U_\beta$*. Moreover, this family of adjunction isomorphisms is natural in* $\alpha$ *and* $\beta$.
5. *Three coherence conditions between these identity and composition isomorphisms are satisfied.*