Homotopy Theory in Type Theory

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Joint work with Eric Finster, Kuen-Bang Hou (Favonia), Michael Shulman
Homotopy Theory

A branch of topology, the study of spaces and continuous deformations
Homotopy

Deformation of one path into another

\[ p \]

\[ q \]
Homotopy

Deformation of one path into another
Homotopy

Deformation of one path into another

\[ p \rightarrow q \]

= 2-dimensional path between paths
Homotopy

Deformation of one path into another

\[ \text{p} \rightarrow \text{q} \]

= 2-dimensional *path between paths*

*Homotopy theory* is the study of spaces by way of their paths, homotopies, homotopies between homotopies, ....
Homotopy groups

\(k^{th}\) homotopy group

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<tr>
<th>n-dimensional sphere</th>
<th>(\mathbb{Z})</th>
<th>(\mathbb{Z}_2)</th>
<th>(\mathbb{Z}_{12})</th>
<th>(\mathbb{Z}_2)</th>
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[Image from Wikipedia]
Type Theory

An alternative to set theory, organized around *types*:

- Basic data types ($\mathbb{N}$, $\mathbb{Z}$, booleans, lists, …)

- Functions

  \[
  \text{double} : \mathbb{N} \to \mathbb{N} \\
  \text{double 0} = 0 \\
  \text{double (n +1)} = \text{double n} + 2
  \]

- Unifies sets and logic
Propositions as Types

1. A proposition is represented by a type
2. A proof is represented by an element of that type

∀x: ℕ. double(x) = 2*x

type of proofs of equality
Propositions as Types

1. A proposition is represented by a type
2. A proof is represented by an element of that type

\[ \forall x : \mathbb{N}.\ double(x) = 2 \times x \]

*type of proofs of equality*
Propositions as Types

1. A proposition is represented by a type
2. A proof is represented by an element of that type

\[ \forall x: \mathbb{N}. \text{double}(x) = 2x \]

\[ \text{proof} : 0 = \text{reflexivity} \]

\[ \text{proof} (n +1) = \ldots \]
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\[ \text{proof } (n +1) = \ldots \]

*proof by case analysis represented by a function defined by cases*
Type are sets?

Traditional view:

\[
\begin{array}{l}
\text{type theory} \\
<\text{element}> : <\text{type}> \\
<\text{elem}_1> = <\text{elem}_2>
\end{array}
\]

\[
\begin{array}{l}
\text{set theory} \\
x \in S \\
x = y
\end{array}
\]

In set theory, an equation is a \textit{proposition}: we don’t ask \textit{why} \(1+1=2\)
Type are sets?

Traditional view:

<table>
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<td><code>&lt;element&gt;</code> : <code>&lt;type&gt;</code></td>
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In set theory, an equation is a *proposition*: we don’t ask *why* \(1+1=2\)

In type theory, an equation has a `<proof>`
Homotopy Type Theory

category theory  type theory  homotopy theory
Types are \( \infty \)-groupoids

[Hofmann, Streicher, Awodey, Warren, Voevodsky, Lumsdaine, Gambino, Garner, van den Berg]
Types are $\infty$-groupoids

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Types are ∞-groupoids

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Homotopy Type Theory

new principles

category theory

type theory

new proofs

homotopy theory
Computer-checked proofs

Your proof

Homotopy Type Theory

Proof checker

Correct!
Incorrect
Synthetic vs Analytic

**Synthetic geometry (Euclid)**

POSTULATES.

I. Let it be granted that a straight line may be drawn from any one point to any other point.

II. That a terminated straight line may be produced to any length in a straight line.

III. And that a circle may be described from any centre, at any distance from that centre.

**Analytic geometry (Descartes)**

![Image from Wikipedia]
Synthetic vs Analytic

Synthetic geometry (Euclid)

Postulates.

I. Let it be granted that a straight line may be drawn from any one point to any other point.
II. That a terminated straight line may be produced to any length in a straight line.
III. And that a circle may be described from any centre, at any distance from that centre.

Classical homotopy theory is analytic:

• a space is a set of points equipped with a topology
• a path is a map $[0,1] \rightarrow X$
Synthetic homotopy theory

<table>
<thead>
<tr>
<th>homotopy theory</th>
<th>type theory</th>
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<tbody>
<tr>
<td>space</td>
<td>\langle\text{type}\rangle</td>
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<tr>
<td>points</td>
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<tr>
<td>paths</td>
<td>\langle\text{proof}\rangle : \langle\text{elem}_1\rangle = \langle\text{elem}_2\rangle</td>
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<td>homotopies</td>
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A path is not a map $[0,1] \to X$; it is a basic notion.
Spaces as types
Spaces as types

a space is a type $A$
Spaces as types

a space is a type $A$

points are elements
$M : A$
Spaces as types

A space is a type $A$

Points are elements $M : A$

Paths are proofs of equality $\alpha : M =_A N$
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a space is a type $A$

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paths are \textit{proofs of equality} $\alpha : M =_A N$

path operations
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a space is a type $A$

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path operations

$id : M = M \ (\text{refl})$
Spaces as types

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Path operations

$\text{id} : M = M \ (\text{refl})$

$\alpha^{-1} : N = M \ (\text{sym})$
Spaces as types

a space is a type $A$

points are elements $M : A$

paths are proofs of equality $\alpha : M =_A N$

path operations

$id : M = M$ (refl)

$\alpha^{-1} : N = M$ (sym)

$\beta \circ \alpha : M = P$ (trans)
Spaces as types

A space is a type $A$.

Points are elements $M : A$.

Paths are proofs of equality $\alpha : M =_A N$.

Path operations:
- $\text{id} : M = M$ (refl)
- $\alpha^{-1} : N = M$ (sym)
- $\beta \circ \alpha : M = P$ (trans)

Homotopies:
- $\text{id} \circ \alpha = \alpha$
- $\alpha^{-1} \circ \alpha = \text{id}$
- $\gamma \circ (\beta \circ \alpha) = (\gamma \circ \beta) \circ \alpha$
Spaces as types

a space is a type $A$

points are elements $M : A$

paths are proofs of equality $\alpha : M =_A N$

path operations

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homotopies

$id \circ \alpha = \alpha$
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We can do computer-checked proofs in \textit{synthetic} homotopy theory
We can do computer-checked proofs in *synthetic* homotopy theory

*Proofs are *constructive*: can run them
We can do computer-checked proofs in **synthetic** homotopy theory

- Proofs are *constructive*: can run them
- Results apply in a variety of settings, from simplicial sets (hence topological spaces) to Quillen model categories and ∞-topoi
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We can do computer-checked proofs in **synthetic** homotopy theory

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- New type-theoretic proofs/methods

*work in progress*
Some results

<table>
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<tr>
<th>Homotopy Theoretic</th>
<th>Type Theoretic</th>
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<td>$\pi_1(S^1)$</td>
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Some results

Homotopy Theoretic  Type Theoretic

$\pi_1(S^1)$  $\pi_1(S^1)$
Some results

Homotopy Theoretic | Type Theoretic

$\pi_1(S^1)$ | $\pi_1(S^1)$

Hopf fibration & $\pi_2(S^2)$
Some results

Homotopy Theoretic          Type Theoretic

$\pi_1(S^1)$                  $\pi_1(S^1)$

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Some results

**Homotopy Theoretic**

- $\pi_1(S^1)$
- Hopf fibration & $\pi_2(S^2)$

**Type Theoretic**

- $\pi_1(S^1)$
- $\pi_2(S^2)$
- $\pi_n(S^n)$
Some results

Homotopy Theoretic

\[ \pi_1(S^1) \]

Hopf fibration

& \ \pi_2(S^2)

Freudenthal

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$\pi_n(S^n)$
Outline

1. $\pi_1(S^1) = \mathbb{Z}$

2. The Hopf fibration

3. Connectedness and Freudenthal Suspension
Outline

1. $\pi_1(S^1) = \mathbb{Z}$

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3. Connectedness and Freudenthal Suspension
Higher inductive types

Circle is *inductively generated* by

![Diagram of loop and base]
Higher inductive types

Circle is *inductively generated* by

\[
\begin{align*}
\text{base} &: \text{Circle} \\
\text{loop} &: \text{base} = \text{base}
\end{align*}
\]
Circle is \textit{inductively generated} by

\begin{itemize}
  \item \textbf{point} \quad base : \text{Circle}
  \item \textbf{loop} : \text{base} = \text{base}
\end{itemize}
Higher inductive types

Circle is \textit{inductively generated} by

- **point** \( \text{base} : \text{Circle} \)
- **path** \( \text{loop} : \text{base} = \text{base} \)
Higher inductive types

Circle is \textit{inductively generated} by

\begin{itemize}
  \item \textbf{point} \hspace{1cm} base : \text{Circle}
  \item \textbf{path} \hspace{1cm} loop : base = base
\end{itemize}

Free $\infty$-groupoid with these generators

\begin{align*}
  \text{id} & \quad \text{inv} : \text{loop} \circ \text{loop}^{-1} = \text{id} \\
  \text{loop}^{-1} & \quad \ldots \\
  \text{loop} \circ \text{loop} & 
\end{align*}
Higher inductive types

Circle recursion:
function Circle → X determined by
base' : X
loop' : base' = base'
Higher inductive types

**Circle recursion:**
function Circle \( \to X \) determined by

- base’ : X
- loop’ : base’ = base’

**Circle induction:** To prove a predicate \( P \) for all points on the circle, suffices to prove \( P(\text{base}) \), continuously in the loop
Definition. $\Omega(S^1)$ is the space of loops at base i.e. the type (base = base)
**Fundamental group of circle**

**Definition.** $\Omega(S^1)$ is the **space** of loops at base
i.e. the type $(\text{base} = \text{base})$

**Theorem.** $\Omega(S^1)$ is equivalent to $\mathbb{Z}$,
by a map that sends $0$ to $+$
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**Proof:** two mutually inverse functions

- $\text{winding} : \Omega(S^1) \to \mathbb{Z}$
- $\text{loop}^n : \mathbb{Z} \to \Omega(S^1)$
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\[
\text{loop}^n : \mathbb{Z} \to \Omega(S^1)
\]

**Corollary:** $\pi_1(S^1)$ is isomorphic to $\mathbb{Z}$
$\pi_k(S^1)$ trivial otherwise

0-truncation (set of connected components) of $\Omega(S^1)$
Universal Cover

\[ w : \Omega(S^1) \rightarrow \mathbb{Z} \]

defined by \textbf{lifting} a loop to the cover, and giving the other endpoint of 0
Universal Cover

lifting is functorial

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lifting is functorial

lifting loop adds 1
Universal Cover

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lifting is functorial
lifting loop adds 1
lifting loop\(^{-1}\) subtracts 1
Universal Cover

\[ w : \Omega(S^1) \to \mathbb{Z} \]

defined by \textbf{lifting} a loop to the cover, and giving the other endpoint of 0

\textbf{Example:}

\[ w(\text{loop \ o \ loop}^{-1}) = 0 + 1 - 1 = 0 \]

lifting is functorial
lifting loop adds 1
lifting loop\(^{-1}\) subtracts 1
Fibration = Family of types

Fibration (classically):
map $p : E \rightarrow B$ such that
any path from $p(e)$ to $y$
lifts to a path in $E$ from $e$
to some point in $p^{-1}(y)$
Fibration = Family of types

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Family of types \((E(x))_{x:B}\)

- *Fibers:* \( E(b) \) is a type for all \( b:B \)
- *transport:* equivalence \( E(b_1) \simeq E(b_2) \) for all \( p:b_1 =_B b_2 \)
Fibration = Family of types

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**Family of types** \( (E(x))_{x:B} \)

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Fibration (classically):
map \( p : E \rightarrow B \) such that
any path from \( p(e) \) to \( y \)
lifts to a path in \( E \) from \( e \)
to some point in \( p^{-1}(y) \)

Family of types \( (E(x))_{x : B} \)

- **Fibers**: \( E(b) \) is a type for all \( b : B \)
- **transport**: equivalence \( E(b_1) \simeq E(b_2) \) for all \( p : b_1 =_B b_2 \)

sends \( e \in E(x) \) to other endpoint of lifting of \( p \)
Universal Cover

family of types \((\text{Cover}(x))_{x:S^1}\)
Universal Cover

**family of types** \((\text{Cover}(x))_{x:S^1}\)

By circle recursion, it suffices to give

- Fiber over base: \(\mathbb{Z}\)

- Equivalence \(\mathbb{Z} \cong \mathbb{Z}\) as lifting of loop: successor
Universal Cover

family of types \((\text{Cover}(x))_{x:S^1}\)

By circle recursion, it suffices to give

- Fiber over base: \(\mathbb{Z}\)
- Equivalence \(\mathbb{Z} \simeq \mathbb{Z}\) as lifting of loop: uses univalence

successor
Universal Cover

family of types \((\text{Cover}(x))_{x:S^1}\)

By circle recursion, it suffices to give

* Fiber over base: \(\mathbb{Z}\)

* Equivalence \(\mathbb{Z} \cong \mathbb{Z}\) as lifting of loop:

  \(\text{successor}\)

Defining equations:

\[
\text{Cover}(\text{base}) := \mathbb{Z}
\]

\[
\text{transport}_{\text{Cover}}(\text{loop}) := \text{successor}
\]
Winding number

\[ w : \Omega(S^1) \to \mathbb{Z} \]

\[ w(p) = \text{transport}_{\text{cover}}(p, 0) \]
Winding number

\[ w : \Omega(S^1) \to \mathbb{Z} \]

\[ w(p) = \text{transport}_{\text{cover}}(p, 0) \]

\[ w(\text{loop}^{-1} \circ \text{loop}) \]

lift p to cover, starting at 0
Winding number

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\[ w(p) = \text{transport}_{\text{Cover}}(p, 0) \]

\[ w(\text{loop}^{-1} \circ \text{loop}) = \text{transport}_{\text{Cover}}(\text{loop}^{-1} \circ \text{loop}, 0) \]

lift \( p \) to cover, starting at 0
Winding number

\[ w : \Omega(S^1) \to \mathbb{Z} \]

\[ w(p) = \text{transport}_{\text{Cover}}(p,0) \]

\[ w(\text{loop}^{-1} \circ \text{loop}) = \text{transport}_{\text{Cover}}(\text{loop}^{-1} \circ \text{loop}, 0) = \text{transport}_{\text{Cover}}(\text{loop}^{-1}, \text{transport}_{\text{Cover}}(\text{loop}, 0)) \]
Winding number

\[ w : \Omega(S^1) \to \mathbb{Z} \]

\[ w(p) = \text{transport}^{\text{Cover}}(p, 0) \]

\[ w(\text{loop}^{-1} \circ \text{loop}) \]
\[ = \text{transport}^{\text{Cover}}(\text{loop}^{-1} \circ \text{loop}, 0) \]
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Winding number

\[ w : \Omega(\mathbb{S}^1) \to \mathbb{Z} \]

\[ w(p) = \text{transport}_{\text{Cover}}(p, 0) \]

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\[ = \text{transport}_{\text{Cover}}(\text{loop}^{-1}, 1) \]
\[ = 0 \]
Fundamental group of the circle

The book

Computer-checked
Outline

1. $\pi_1(S^1) = \mathbb{Z}$

2. The Hopf fibration

3. Connectedness and Freudenthal Suspension
The Hopf fibration is a fibration with

- base $S^2$
- fiber $S^1$
- total space $S^3$
The Hopf fibration

The Hopf fibration is a fibration with

- base $S^2$
- fiber $S^1$
- total space $S^3$

The Hopf fibration is a family of circles, parametrized by $S^2$ and whose “union” is $S^3$. 
$\pi_1(S^1) = \mathbb{Z}$

The Hopf fibration
The Hopf fibration

The spheres

**Definition**

The suspension of a space $A$ (denoted $\Sigma A$) is generated by

- Two points $n, s : \Sigma A$
- For every $a : A$, a path $m(a) : n =_{\Sigma A} s$

**Definition**

$$S^{n+1} := \Sigma S^n$$
Fibrations over $\mathbb{S}^2$

A fibration over $\mathbb{S}^2$ is given by

- a space $A$ (over $n$)
Fibrations over $\mathbb{S}^2$

A fibration over $\mathbb{S}^2$ is given by

- a space $A$ (over $n$)
- a space $B$ (over $s$)
A fibration over $\mathbb{S}^2$ is given by

- a space $A$ (over $n$)
- a space $B$ (over $s$)
- a “circle of equivalences” between $A$ and $B$ (over $m$)

\[ \iff a \text{ function } e : \mathbb{S}^1 \to (A \simeq B) \]
\[ \iff \text{ for every } x : \mathbb{S}^1, \text{ an equivalence } e_x : A \simeq B \]
The Hopf fibration in HoTT

A fibration over $S^2$ with fiber $S^1$ and total space $S^3$?

Left to do:
- Define the rotation of angle $x$
- Prove that the total space is $S^3$
The Hopf fibration in HoTT

A fibration over $S^2$ with fiber $S^1$ and total space $S^3$?

- $S^1$ over $n$
- $S^1$ over $s$
- for $x : S^1$, the equivalence $e_x : S^1 \simeq S^1$ is the “rotation of angle” $x$
The Hopf fibration in HoTT

A fibration over $S^2$ with fiber $S^1$ and total space $S^3$?

- $S^1$ over $n$
- $S^1$ over $s$
- for $x : S^1$, the equivalence $e_x : S^1 \sim S^1$ is the “rotation of angle” $x$

Left to do:

- Define the rotation of angle $x$
- Prove that the total space is $S^3$
Rotations of $\mathbb{S}^1$

We want

$$e : \mathbb{S}^1 \to (\mathbb{S}^1 \simeq \mathbb{S}^1)$$
Rotations of $\mathbb{S}^1$

We want

$$e : \mathbb{S}^1 \rightarrow (\mathbb{S}^1 \simeq \mathbb{S}^1)$$

By definition of $\mathbb{S}^1$, we need

- an equivalence $e_{\text{base}} : \mathbb{S}^1 \simeq \mathbb{S}^1$
- a homotopy $e(\text{loop}) : e_{\text{base}} = e_{\text{base}}$
Rotations of $\mathbb{S}^1$

We want

$$e : \mathbb{S}^1 \to (\mathbb{S}^1 \simeq \mathbb{S}^1)$$

By definition of $\mathbb{S}^1$, we need

- an equivalence $\text{id}_{\mathbb{S}^1} : \mathbb{S}^1 \simeq \mathbb{S}^1$
- a homotopy $e(\text{loop}) : \text{id}_{\mathbb{S}^1} = \text{id}_{\mathbb{S}^1}$
Rotations of $\mathbb{S}^1$

We want

$$e : \mathbb{S}^1 \to (\mathbb{S}^1 \simeq \mathbb{S}^1)$$

By definition of $\mathbb{S}^1$, we need

- an equivalence $\text{id}_{\mathbb{S}^1} : \mathbb{S}^1 \simeq \mathbb{S}^1$
- a homotopy $e(\text{loop}) : \text{id}_{\mathbb{S}^1} = \text{id}_{\mathbb{S}^1}$

$e(\text{loop})$ is the homotopy “turning once around the circle”.

[Diagram of a loop around a circle with a base point labeled as the loop's starting point.]
Homotopy turning once around the circle

A homotopy $\text{id}_{S^1} = \text{id}_{S^1}$ $\iff$ for every $x : S^1$, a path $x = x$
The Hopf fibration

Homotopy turning once around the circle

A homotopy $\text{id}_{S^1} = \text{id}_{S^1}$ $\iff$ for every $x : S^1$, a path $x = x$

We need:

- a path
  
  $p : \text{base} = \text{base}$

- a (2-dimensional) path
  
  $q : p \cdot \text{loop} = \text{loop} \cdot p$
The Hopf fibration

Homotopy turning once around the circle

A homotopy $\text{id}_{S^1} = \text{id}_{S^1}$ $\iff$ for every $x : S^1$, a path $x = x$

We need:

- a path
  $$\text{loop} : \text{base} = \text{base}$$
- a (2-dimensional) path
  $$q : \text{loop} \cdot \text{loop} = \text{loop} \cdot \text{loop}$$
A homotopy $\text{id}_{S^1} = \text{id}_{S^1} \iff$ for every $x : S^1$, a path $x \equiv x$

We need:

- a path

\[ \text{loop} : \text{base} \equiv \text{base} \]

- a (2-dimensional) path

\[ \text{refl}_{\text{loop}} : \text{loop} \cdot \text{loop} \equiv \text{loop} \cdot \text{loop} \]
The Hopf fibration

Total space

We just constructed a fibration with
- base $S^2$
- fiber $S^1$

What is the total space?
Homotopy pushouts

Given a span

\[
\begin{array}{c}
Y & \xleftarrow{f} & X & \xrightarrow{g} & Z
\end{array}
\]

Definition

The **homotopy pushout** $Y \sqcup^X Z$ is the space generated by

- For all $y : Y$, a point $l(y) : Y \sqcup^X Z$
- For all $z : Z$, a point $r(z) : Y \sqcup^X Z$
- For all $x : X$, a path $g(x) : l(f(x)) = r(g(x))$

The suspension of $A$ is the homotopy pushout of

\[
1 \xleftarrow{} A \xrightarrow{} 1
\]
By gluing/descent/flattening, the total space is the homotopy pushout of:

\[
S^1 \xleftarrow{e} S^1 \times S^1 \xrightarrow{p_2} S^1
\]
Total space

By gluing/descent/flattening, the total space is the homotopy pushout of:

\[
\begin{align*}
S^1 & \leftarrow^e S^1 \times S^1 \rightarrow^{p_2} S^1 \\
\end{align*}
\]

This span is equivalent to the following:

\[
\begin{align*}
S^1 & \leftarrow^{p_1} S^1 \times S^1 \rightarrow^{p_2} S^1 \\
\end{align*}
\]

whose total space is \( S^1 \ast S^1 \)
The join of $A$ and $B$ is the homotopy pushout of

$$A \leftarrow A \times B \rightarrow B$$

**Definition**

The join of $A$ and $B$ is the homotopy pushout of

$$A \leftarrow A \times B \rightarrow B$$
Join

**Definition**

The *join* of $A$ and $B$ is the homotopy pushout of

$$
\begin{array}{ccc}
A & \xleftarrow{p_1} & A \times B & \xrightarrow{p_2} & B \\
& & & & \\
\end{array}
$$

We have

$$
S^0 \star A = \Sigma A
$$

$$(A \star B) \star C = A \star (B \star C)$$
$$\pi_1(S^1) = \mathbb{Z}$$

The Hopf fibration

**Total space**

\[ S^1 \ast S^1 = (\Sigma S^0) \ast S^1 \]
\[ = (S^0 \ast S^0) \ast S^1 \]
\[ = S^0 \ast (S^0 \ast S^1) \]
\[ = \Sigma(\Sigma S^1) \]
\[ = S^3 \]
Total space

\[ S^1 \star S^1 = (\Sigma S^0) \star S^1 \]

\[ = (S^0 \star S^0) \star S^1 \]

\[ = S^0 \star (S^0 \star S^1) \]

\[ = \Sigma (\Sigma S^1) \]

\[ = S^3 \]

We have the Hopf fibration in homotopy type theory.
The Hopf fibration

Long exact sequence

Long exact sequence of homotopy groups of the Hopf fibration:

\[\pi_4(S^1) \longrightarrow \pi_4(S^3) \longrightarrow \pi_4(S^2)\]

\[\pi_3(S^1) \longrightarrow \pi_3(S^3) \longrightarrow \pi_3(S^2)\]

\[\pi_2(S^1) \longrightarrow \pi_2(S^3) \longrightarrow \pi_2(S^2)\]

\[\pi_1(S^1) \longrightarrow \pi_1(S^3) \longrightarrow \pi_1(S^2)\]
The Hopf fibration

Long exact sequence of homotopy groups of the Hopf fibration:

\[ \ldots \longrightarrow \pi_4(S^3) \longrightarrow \pi_4(S^2) \longrightarrow \pi_3(S^3) \longrightarrow \pi_3(S^2) \longrightarrow \pi_2(S^3) \longrightarrow \pi_2(S^2) \longrightarrow \pi_1(S^3) \longrightarrow \pi_1(S^2) \longrightarrow 0 \]
Long exact sequence

Long exact sequence of homotopy groups of the Hopf fibration:

\[ \cdots \xrightarrow{} 0 \xrightarrow{} \pi_4(S^3) \xrightarrow{} \pi_4(S^2) \xrightarrow{} 0 \xrightarrow{} \pi_3(S^3) \xrightarrow{} \pi_3(S^2) \xrightarrow{} 0 \xrightarrow{} \pi_2(S^2) \xrightarrow{} 0 \xrightarrow{} \mathbb{Z} \xrightarrow{} 0 \xrightarrow{} 0 \xrightarrow{} \cdots \]
Long exact sequence

Long exact sequence of homotopy groups of the Hopf fibration:

\[ \cdots \]

\[ 0 \overset{\sim}{\longrightarrow} \pi_4(S^3) \overset{\sim}{\longrightarrow} \pi_4(S^2) \]

\[ 0 \overset{\sim}{\longrightarrow} \pi_3(S^3) \overset{\sim}{\longrightarrow} \pi_3(S^2) \]

\[ 0 \overset{\sim}{\longrightarrow} 0 \overset{\sim}{\longrightarrow} \pi_2(S^2) \]

\[ \mathbb{Z} \overset{\sim}{\longrightarrow} 0 \overset{\sim}{\longrightarrow} 0 \]
Homotopy groups

Theorem

We have

\[ \pi_2(S^2) = \mathbb{Z} \]

\[ \pi_k(S^2) = \pi_k(S^3) \text{ for } k \geq 3 \]

In particular

Theorem

Assuming \( \pi_3(S^3) = \mathbb{Z} \)

\[ \pi_3(S^2) = \mathbb{Z} \]
\[ \pi_1(S^1) = \mathbb{Z} \]

\[ \pi_4(S^3) \]

**Theorem**

There exists a natural number \( n \) such that \( \pi_4(S^3) \cong \mathbb{Z}/n\mathbb{Z} \).
The Hopf fibration

\[ \pi_4(S^3) \]

**Theorem**

There exists a natural number \( n \) such that \( \pi_4(S^3) \cong \mathbb{Z}/n\mathbb{Z} \).

- Classical mathematics: cannot compute \( n \), unless the proof is nice enough
The Hopf fibration

\[ \pi_4(S^3) \]

**Theorem**

There exists a natural number \( n \) such that \( \pi_4(S^3) \cong \mathbb{Z}/n\mathbb{Z} \).

- Classical mathematics: cannot compute \( n \), unless the proof is nice enough
- Constructive mathematics: disallow the axiom of choice and excluded middle \( \implies \) every proof is nice enough

\[ \pi_1(S^1) = \mathbb{Z} \]
The Hopf fibration

\[ \pi_1(S^1) = \mathbb{Z} \]

\[ \pi_4(S^3) \]

**Theorem**

*There exists a natural number \( n \) such that \( \pi_4(S^3) \cong \mathbb{Z}/n\mathbb{Z} \).*

- Classical mathematics: cannot compute \( n \), unless the proof is nice enough
- Constructive mathematics: disallow the axiom of choice and excluded middle \( \iff \) every proof is nice enough

In this case we can compute the value of \( n \) and get 2*

*work in progress*
1. $\pi_1(S^1) = \mathbb{Z}$

2. The Hopf fibration

3. Connectedness and Freudenthal Suspension
Part III: Freudenthal and friends

1. Truncatedness

2. Connectedness

3. Freudenthal Suspension Theorem
Truncatedness

Definition

A type \( X \) is \( n \)-truncated (or an \( n \)-type) if, by induction on \( n \geq -2 \):

- \( n = -2 \): if \( X \) is contractible, i.e. \( X \simeq 1 \);
- \( n > -2 \): if each path space \( (x =_X x') \) of \( X \) is \( (n - 1) \)-truncated.

Proposition

Suppose \( X \) is \( n \)-truncated, for \( n \geq -1 \). Then \( \pi_k(X, x_0) \simeq 1 \), for all \( k > n \) and \( x_0 : X \).

[In \textbf{Top} and \textbf{SSet}, the converse holds; but not in all classical settings, cf. Whitehead’s theorem and hypercompleteness.]
Truncations

Definition

For any type $X$, and $n \geq -1$, the $n$-truncation $\tau_n X$ is the higher inductive type generated by:

- for $x : X$, an element $[x]_n : \tau_n X$;
- for $f : S^{n+1} \to \tau_n X$, and $t : S^{n+1}$, a path $f(t) = f(0)$.

Proposition

$\tau_n X$ is the free $n$-truncated type on $X$: any $f : X \to Y$, with $Y$ $n$-truncated, factors uniquely through $\tau_n X$.

[Classically: iteratively glue cells on to $X$ to kill homotopy in dimensions $> n$.]
**Definition**

$X$ is *$n$-connected* if $\tau_{n+1}X$ is contractible.

<table>
<thead>
<tr>
<th>Proposition</th>
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<tbody>
<tr>
<td>TFAE:</td>
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<tr>
<td>▶ $X$ is $n$-connected;</td>
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<tr>
<td>▶ <em>every map from $X$ to an $n$-type is constant</em>;</td>
</tr>
<tr>
<td>▶ <em>when $n \geq 0$</em> $\pi_k(X, x_0) \simeq 1$, for all $k \leq n$ and $x_0 : X$.</td>
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</table>

Connectedness (trivial low homotopy groups) is dual to truncatedness (trivial high homotopy groups).
Connectedness (of maps)

**Definition**

\( f : A \to B \) is \textit{n-connected} if each (homotopy) fiber \( f^{-1}(b) \) is \( n \)-connected. (Warning: indexing conventions vary by \( \pm 1 \).)

**Proposition**

\textit{TFAE:}

- \( f \) is \( n \)-connected;
- \( f \) is weakly (or strongly) orthogonal to maps with \( n \)-truncated fibers;
- \( f \) is equivalent to the inclusion of \( A \) into some extension by cells of dimensions \( \geq n \).
Additivity of connectedness

Lemma (Wedge-product connectedness)

Suppose \((X, x_0)\) is \(i\)-connected, \((Y, y_0)\) is \(j\)-connected. Then the inclusion \(X \sqcup_1 Y \hookrightarrow X \times Y\) is \((i + j)\)-connected.

Type-theoretically: to define a function of two variables \(f(x, y)\) into an \((i + j)\)-type, enough to define in the cases \(f(x_0, y)\) and \(f(x, y_0)\), agreeing in the case \(f(x_0, y_0)\).
**Freudenthal**

**Definition**

Recall: the **suspension** $\Sigma X$ is generated by

- $N, S : \Sigma X$;
- for each $x : X$, a path $m(x) : N \to \Sigma X \supset S$.

**Theorem (Freudenthal Suspension Theorem)**

*Suppose $(X, x_0)$ is $n$-connected. Then the canonical map $X \to \Omega(\Sigma X, N)$ is $2n$-connected.*

Idea: want $X \to \Omega(\Sigma X, N)$ to be an equivalence. Generally (e.g. for $\Sigma S^1 \simeq S^2$) it isn’t; but within a certain dimension range, it is.

Important application: stable homotopy groups of spheres.
Proof: weak Freudenthal

For now, prove a weaker statement. (Same approach, with more work, yields full FST.)

**Theorem (Weak Freudenthal)**

*Suppose \((X, x_0)\) is \(n\)-connected. Then the canonical map \(\tau_{2n}(X) \to \tau_{2n}\Omega(\Sigma X, N)\) is an equivalence.*

**Proof.**

Heuristic: to prove a result of the form \(X \approx \Omega(Y, y_0)\), generalise \(X\) to a dependent type \(\tilde{X}_y\) over \(y : Y\), with \(\tilde{X}_{y_0} \simeq X\), and prove \(\tilde{X}_y \simeq (y_0 =_Y Y)\) for all \(y : Y\).

So: define type \(\tilde{X}_y\) depending on \(y : \Sigma X\), and maps \(\bar{m}_y : \tilde{X}_y \to \tau_{2n}(N = y)\), using universal property of \(\Sigma X\).
Weak Freudenthal, cont’d

Proof.

To give \( \bar{X}_y, \bar{m}_y \) for all \( y : \Sigma X \), need:

- types and maps \( \bar{m}_N : \bar{X}_N \rightarrow \tau_{2n}(N = N) \), and
  \( \bar{m}_S : \bar{X}_S \rightarrow \tau_{2n}(N = S) \);

- transport equivalences \( \text{transport}_{\bar{X}}m(x_1) : \bar{X}_N \rightarrow \bar{X}_S \), for each \( x_1 : X \), commuting with \( \bar{m}_N, \bar{m}_S \).

over \( S \):

\[ \bar{m}_S := \tau_{2n}(m) : \tau_{2n}(X) \rightarrow \tau_{2n}(N = S) \]

over \( N \):

\[ \bar{m}_N := \tau_{2n}(x \mapsto m(x) \circ m(x_0)^{-1}) : \tau_{2n}(X) \rightarrow \tau_{2n}(N = N) \]

and over \( m(x) \), need to define for each \( x_1 : X \) the action

\( \text{transport}_{\bar{X}}(m(x), -) : \bar{X}_N \rightarrow \bar{X}_S \).
Weak Freudenthal, cont’d

Proof.

\[ \ldots \text{transport over } m(x_1) : \text{need to give, for each } x_1 : X \text{ and } z : \bar{X}_N = \tau_{2n}(X), \text{some element of } \bar{X}_S = \tau_{2n}(X). \]

Since RHS is 2n-truncated, may assume \( z \) is of form \([x_2]\), some \( x_2 : X \). Also, by wedge-product connectedness lemma, enough to assume one of \( x_1, x_2 \) is \( x_0 \). So: when \( x_1 = x_0 \), return \([x_2]\).

When \( x_2 = x_0 \), return \([x_1]\). (Check: when \( x_1 = x_2 = x_0 \), these agree)

(Roughly: defining a multiplication \( X \times \tau_{2n}(X) \rightarrow \tau_{2n}(X) \), with \( x_0 \) as a two-sided unit.)

So: have \( \bar{m}_y : \bar{X}_y \rightarrow (N = y) \), for all \( y : \Sigma X \).

Define converse \( \bar{n}_y : (N = y) \rightarrow \bar{X}_y \) by \( n_y(p) := \text{transport}_{\bar{X}}[x_0] \).

Not hard to prove \( \bar{m}, \bar{n} \) mutually inverse; so, each \( \bar{m}_y \) is an equivalence, as desired. \( \square \)
Consequences

From (weak) Freudenthal, immediately have:

Corollary (Homotopy groups of spheres stabilise)

\[ \pi_{n+k}(S^n) \simeq \pi_{n+1+k}(S^{n+1}), \text{ for } n \geq k + 2. \]

In particular,

Corollary

\[ \pi_n(S^n) \simeq \mathbb{Z}, \text{ for all } n \geq 1. \]

Proof.

- \( n = 1 \): by universal cover.
- \( n = 2 \): by LES of Hopf fibration.
- \( n \geq 2 \): by stabilisation.
\[ \pi_k(S^n) \] in HoTT

\[ k^{\text{th}} \text{ homotopy group} \]

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<tr>
<th>n-dimensional sphere</th>
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More results
James construction

Refinement of Freudenthal: describes $\Omega(\Sigma X)$ precisely, via a filtration.

**Theorem**

Suppose $(X, x_0)$ is $n$-connected, for $n \geq 0$. There is a sequence

$$1 \longrightarrow X \longrightarrow J_2(X) \longrightarrow J_3(X) \longrightarrow J_4(X) \longrightarrow \cdots$$

with the maps having respective connectivities $(n-1)$, $2n$, $(3n+1)$, \ldots, and such that $J_\infty(X) := \varinjlim J_n(X) \simeq \Omega(\Sigma X)$.

Conceptually, $J_\infty(X)$ is the free monoid on $X$; as $X$ is connected, this is the free group on $X$. 
Blakers–Massey

Generalization of Freudenthal: describes path spaces in pushouts.

Theorem (Blakers–Massey theorem)

Suppose given maps $f, g$ as below, with $f$ $i$-connected, $g$ $j$-connected.

$$
\begin{array}{c}
Z \\ \downarrow f \\
X \\
\end{array} \quad \begin{array}{c}
g \\ \downarrow \mathrm{inr} \\
Y \\
\end{array} \quad \begin{array}{c}
X \\
\downarrow \mathrm{inl} \\
X \sqcup_Z Y \\
\end{array}
$$

Then for all $x : X, y : Y$, the canonical map $Z_{x,y} \to (\mathrm{inl} \ x = \mathrm{inr} \ y)$ is $(i + j)$-connected.
Another tool for pushouts of types:

**Theorem (van Kampen theorem)**

For any pointed maps $f : Z \to X$ and $g : Z \to Y$, with $Z$ 0-connected, the fundamental group of the pushout of $f$ and $g$ is the amalgamated free product (pushout of groups) of $\pi_1(X)$ and $\pi_1(Y)$ over $\pi_1(Z)$:

$$\pi_1(X \sqcup_Z Y) \simeq \pi_1(X) \ast_{\pi_1(Z)} \pi_1(Y).$$

*Can also be generalised to non-connected $Z$.***
The (beautiful) classical theory of covering spaces transfers straightforwardly. In particular:

<table>
<thead>
<tr>
<th>Definition</th>
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<td>A <strong>covering space</strong> of a connected type $X$ is a dependent family of 0-types over $X$.</td>
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</table>

<table>
<thead>
<tr>
<th>Theorem</th>
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<tbody>
<tr>
<td><strong>Covering spaces of</strong> $X$ <strong>correspond to sets with an action of</strong> $\pi_1(X)$ <strong>.</strong></td>
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</tbody>
</table>
Eilenberg–Mac Lane spaces; cohomology

Eilenberg–Mac Lane spaces of Abelian groups can be constructed as HIT’s:

**Theorem**

For any \((n\text{-truncated})\) Abelian group \(G\) and natural number \(n > 0\), there is a type \(K(G, n)\) such that \(\pi_n(K(G, n)) \simeq G\), and \(\pi_k(K(G, n)) \simeq 1\) for \(k \neq n\).

These (and other spectra) can be used to define cohomology of types.
Conclusion
We can do computer-checked proofs in \textit{synthetic} homotopy theory
\[ \pi_1(S^1) = \mathbb{Z} \]

\[ \pi_{k<n}(S^n) = 0 \]
April 11, 2013

\[ \pi_1(S^1) = \mathbb{Z} \]
\[ \pi_{k<n}(S^n) = 0 \]
Hopf fibration
\[ \pi_2(S^2) = \mathbb{Z} \]
\[ \pi_3(S^2) = \mathbb{Z} \]
James Construction
\[ \pi_4(S^3) = \mathbb{Z}? \]
Freudenthal
\[ \pi_n(S^n) = \mathbb{Z} \]
K(G,n)
Cohomology axioms
Blakers-Massey
Van Kampen
Covering spaces
Whitehead for n-types